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A rotated Dynamic Nelson-Siegel model with macro-financial applications

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Abstract

A factor rotation scheme is applied to the well-known Dynamic Nelson-Siegel model facilitating direct parametrization of the short rate process. The model-implied term structure of term premia is derived in closed-form, and macroeconomic variables are included in a Taylor-rule-type fashion. Four empirical experiments are performed on US data covering the period from 1990 to 2014. It is found that macroeconomic variables impact the evolution of the short rate until 2002, after which their effects become insignificant in a statistical sense. The calculated term structure of term premia is robust to the tested parameterzations, and traces out the interest rate cycles present in the data.

Keywords: *Yield Curve Modeling, Dynamic Nelson-Siegel Model, Term Premia, Factor Rotation, Policy Rate, State Space Model.*

JEL: *G1, E4, C5*

Non-technical summary

A new parametrization of the well-known Nelson-Siegel yield curve model-family is derived, in which the underlying factors have interpretations as the short rate, the slope, and the curvature. This contrasts with the traditional Nelson-Siegel model where factors can be interpreted as the level, (minus) the slope and the curvature. One advantage of the proposed new parametrization is that the dynamic behavior of the short rate can be modeled directly and that it can be defined to match any short-term maturity τ_i. This is not possible in the traditional dynamic Nelson-Siegel modeling set-up because the short rate would be calculated as the sum of the first two yield curve factors, and would only be valid for $\tau_i = 0$. Since the short rate process is parameterized directly in the proposed model rotation, it is possible to include macroeconomic variables in a Taylor-rule type fashion, and to obtain a closed-form expression for the model-implied term-premia.

A model evaluation is performed on monthly US data spanning the period from 1990 to 2014, using four different model specifications, and the tested parameterizations comprise macroeconomic variables and time-variability in model parameters.

It is found that the estimated term premia compare well with previous estimates presented in the literature and that there is little added value from allowing parameters to vary over the sample. Furthermore, macroeconomic variables are found to be statistically significant only for the period from 1990 to 2002.

1 Introduction

The Dynamic Nelson-Siegel (DNS) model has become a yardstick for policy-oriented yield curve modeling work in public organizations and central banks (see e.g. BIS (2005) and Diebold and Rudebusch (2013)). A particular trait of the DNS model is the close proximity of its underlying factors to observable quantities: the evolution of the first factor (the level) resembles that of the long end of the yield curve, e.g. the ten year maturity segment; the second factor (minus the slope) is similar to the difference between the short end of the yield curve and the level factor; and the curvature factor, which has an intuitive interpretation as the convergence speed of the short rate to its long-term mean, can be approximated by a weighted average of yields observed at short, medium, and long-term maturities.[1] Although the Nelson-Siegel classification of the underlying

[1] Other existing models also produce yield curve factors that approximate the curve's level, slope and curvature; see e.g. Litterman and Scheinkman (1991), Ang and Piazzesi (2003), Duffie and Kan (1996), Dai and Singleton (2000). However, these models do not explicitly impose a particular interpretation on the extracted factors - it is just observed empirically that this particular factor interpretation fits well.

yield curve factors is useful, it does have some potential drawbacks since it does not facilitate direct parametrization of the short rate process itself. Within the DNS modeling framework, the short rate is defined as the sum of the first two yield curve factors, and for this reason it is, for example, not trivial (i) to identify which time-series process it follows; (ii) to calculate the model implied short rate for maturities different from $\tau_i = 0$; (iii) to embed an empirical monetary policy rule where the short end of the yield curve is directly affected by macro economic variables. Also, the DNS model does not allow for direct calculation of model implied term premia.[2]

To overcome these drawbacks, I propose a version of the DNS model where the yield curve factors are 'rotated' such that the short rate appears explicitly as one of the modeled factors. In the context of the proposed Rotated Dynamic Nelson-Siegel (RDNS) model, it is therefore possible to integrate exogenous variables that are believed to affect the evolution of the short rate (e.g. in the form of a monetary policy rule) and to derive a closed-form expression for the term structure of term premia implied by the model.

It is in principle possible to obtain a multitude of alternative interpretations of the DNS model's underlying yield curve factors by applying different factor rotation schemes. In statistical analysis, rotation schemes are typically implemented to improve the intuitive interpretation of extracted factors; for example, the so-called varimax rotation aims to maximize the loadings of each of the original variables on as few principal components/factors as possible, thus making it easier to give meaning to the observed data (see, Johnson and Wichern (1992)[p.422]). In the current paper, the proposed rotation of the DNS factors allows the extraction of a Short Rate, a Slope, and a Curvature factor from the observed yield curve data. These factors have observational equivalents, as the DNS factors have. The first RDNS factor is a latent yield defined at a pre-selected short-term maturity; the second RDNS factor corresponds to the difference between a long-maturity yield and the short rate factor, and it is therefore equal to the DNS slope factor, just with opposite sign; and the third RDNS factor is similar to the DNS curvature factor.

The RDNS model tested in the current paper is formulated in state-space form and includes (i) time-variability in the time-decay parameter; (ii) a VARMA(1,1) structure for the dynamic evolution of the yield curve factors; (iii) macroeconomic variables affecting the short rate process in the form of an empirical Taylor rule (Taylor (1993)); and (iv) exogenously imposed structural breaks.

[2] Macroeconomic variables have been integrated into the DNS framework in a number of applications; see among others, Diebold, Rudebusch, and Aruoba (2006). However, when macro variables are combined with the level, the slope, and the curvature factors, it is not possible to separately account for how term premia, on the one hand, and the underlying drivers of the yield curve, on the other hand, are affected. The proposed rotation of the DNS model facilitates such a separation.

Using US data, covering the period from 1990 to 2014, it is shown that the derived term-structure of term permia is robust to different model specifications. Furthermore, macroeconomic variables only play a role in determining the evolution of the short rate until 2002, after which their effects on the short rate process disappear. In terms of extracting information about the dynamics of the yield curve factors and term premia, there does not seem to be any added benefit from allowing the model's time-decay parameter to vary over time, or for allowing moving-average components in the equation that characterizes the dynamic behavior of the yield curve factors. In fact, the empirical analysis shows that there is a trade-off between time-variability in the time-decay parameter and the statistical significance of the moving-average parameters. Across the tested parameterizations, the autoregressive parameters for the yield curve factors are all strongly significantly different from zero, and a significant relationship exists between the Curvature and the Short Rate factors.

2 The Rotated Dynamic Nelson-Siegel Model

Denote by $y_t(\tau_i)$ a single yield observation made at time t, for a specific maturity τ_i. The cross-section of yields $y_t(\tau_i)$ $\forall i$ then spans the yield curve observed at t, and the set of yields observed for dates $t = \{1, \ldots, T\}$ then form the yield curve surface, i.e. the evolution of the whole yield curve over time.

According to the parametric description of the yield curve, as suggested by Nelson and Siegel (1987), and extended to a dynamic framework by Diebold and Li (2006), Diebold, Li, and Yue (2007), and Diebold, Rudebusch, and Aruoba (2006), a state-space representation for the yield-curve factor dynamics, and the time t observed yield curve, can be written as:

$$y_t(\tau) = H \cdot \beta_t + e_t \tag{1}$$

$$\beta_t = \mu + F \cdot \beta_{t-1} + v_t \tag{2}$$

The observation equation of the model is shown in (1) and provides the cross-sectional relationship that the model imposes between observed yields (LHS) and the chosen yield curve factor representation, $\beta = [Level,\ -Slope,\ Curvature]$ (RHS). The parametrization of the factor-loading matrix is given by:

$$H = \left[1 \quad \frac{1-exp(-\lambda_t \cdot \tau)}{\lambda_t \cdot \tau} \quad \frac{1-exp(-\lambda_t \cdot \tau)}{\lambda_t \cdot \tau} - exp(-\lambda_t \cdot \tau) \right], \tag{3}$$

where the variable, λ_t, is a time-decay parameter that determines the shape and decay-speed of the slope and curvature loadings, and e_t is an error-term.

The dynamic evolution of the yield curve factors is specified by the transition equation in (2). Here it is written as a vector autoregressive model of order one, VAR(1), but this is naturally not restrictive given the companion-form representation of any VAR(p) model, having $p > 1$. The matrix F collects the autoregressive parameters and v_t is an error-term.

In order to rotate the Nelson-Siegel factors, from $[Level, -Slope, Curvature]$ into $[Short\ Rate, Slope, Curvature]$, the DNS model is augmented with a rotation matrix, denoted by A. This rotation matrix is chosen in such a way that the desired factor-interpretation emerges, and such that $I = A^{-1} \cdot A$, where I is the identity matrix. It is therefore ensured that the RDNS model has exactly the same statistical properties as the DNS model, i.e. it is observational equivalent to the DNS model. Naturally, this equivalence ends when the RDNS model is augmented to include e.g. exogenous variables.

Similar to equations (1) and (2), the RDNS model is written in state space form as:

$$y_t(\tau) = H \cdot A^{-1} \cdot A \cdot \beta_t + e_t \tag{4}$$

$$A \cdot \beta_t = A \cdot \mu + F \cdot A \cdot \beta_{t-1} + A \cdot v_t \tag{5}$$

To simplify the notation let:

$$G = H \cdot A^{-1}, \tag{6}$$

$$\gamma_t = A \cdot \beta_t, \tag{7}$$

$$m = A \cdot \mu, \tag{8}$$

$$z_t = A \cdot v_t, \tag{9}$$

whereby the proposed standard form of the RDNS model is given by:

$$y_t(\tau) = G \cdot \gamma_t + e_t \tag{10}$$

$$\gamma_t = m + F \cdot \gamma_{t-1} + z_t \tag{11}$$

As the last element of the RDNS model, the rotation matrix A, has to be parameterized. Given that a certain economic interpretation of the factors is desired, equation (7) can help to fix ideas. Since A pre-multiplies β, each row of A can be seen as a linear combination of the original DNS factors that provide the desired interpretation of the RDNS factors:

$$\begin{bmatrix} Short\ Rate \\ Slope \\ Curvature \end{bmatrix} = \begin{bmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{bmatrix} \cdot \begin{bmatrix} Level \\ -Slope \\ Curvature \end{bmatrix}, \tag{12}$$

where $a_{(r,c)}$ denotes the row r and column c element of A. If the short rate is defined for maturity $\tau = 0$, then it follows that the rotation matrix has a simple structure as:

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{13}$$

However, it may not be desirable to define the short rate process at the absolute minimum of all possible maturity values. In fact, for policy purposes, and in the context of integrating macroeconomic variables into the model, it may be preferred to work with a short rate defined at the 1 or 3 months maturity segment. When adopting a short rate definition at a maturity higher than zero, the rotation matrix, A, must then be modified to reflect the actual DNS loadings at this chosen maturity, and it will therefore deviate from the simple form shown in (13).

For the purpose of the current application of the RDNS model, the key-maturity (τ_s) around which the rotation is performed, is set to be 3 months. $\tau_s = 3$ months is chosen because this maturity reflects well the stance of monetary policy while escaping institutional features of the money market, in particular liquidity-demand effects, that may affect rates observed at very short maturities.

To derive a general expression for A, one can think of the 'short rate' as the 'long end of the curve' minus the 'slope', plus a small 'curvature correction' because $\tau_s > 0$. It is recalled that in the DNS model the sign of the slope factor is reversed, and, thus, a negative value for the DNS slope factor constitutes an upward sloping yield curve. Therefore, the first row of the general rotation matrix A_{τ_s}, denoted by $A_{(1,1:3),\tau_s}$, is given by:

$$Short\ Rate^{RDNS}_{\tau_s} = a_{(1,1),\tau_s} \cdot Level^{DNS} + a_{(1,2),\tau_s} \cdot Slope^{DNS} + a_{(1,3),\tau_s} \cdot Curvature^{DNS}$$

$$\Downarrow$$

$$A_{(1,1:3),\tau_s} = \begin{bmatrix} a_{(1,1),\tau_s} & a_{(1,2),\tau_s} & a_{(1,3),\tau_s} \end{bmatrix} = \begin{bmatrix} 1 & \frac{1-exp(-\lambda_t \cdot \tau_s)}{\lambda_t \cdot \tau_s} & \frac{1-exp(-\lambda_t \cdot \tau_s)}{\lambda_t \cdot \tau_s} - exp(-\lambda \cdot \tau_s) \end{bmatrix}$$

Following this approach, the second row of A_{τ_s}, denoted by $A_{(2,1:3),\tau_s}$, expresses the linear combination of the DNS factors that form the slope in the context of the proposed RDNS model. The slope can empirically be calculated as difference between the long and short ends of the yield curve, and is therefore:

$$\begin{aligned} A_{(2,1:3),\tau_s} &= \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 1 & \frac{1-exp(-\lambda_t \cdot \tau_s)}{\lambda_t \cdot \tau_s} & \frac{1-exp(-\lambda_t \cdot \tau_s)}{\lambda_t \cdot \tau_s} - exp(-\lambda_t \cdot \tau_s) \end{bmatrix} \\ &= \begin{bmatrix} 0 & -\left(\frac{1-exp(-\lambda_t \cdot \tau_s)}{\lambda_t \cdot \tau_s}\right) & -\left(\frac{1-exp(-\lambda_t \cdot \tau_s)}{\lambda_t \cdot \tau_s} - exp(-\lambda_t \cdot \tau_s)\right) \end{bmatrix}. \end{aligned}$$

Using an empirical definition for the curvature component, as two times a medium maturity yield, minus the long term yield, minus the short term yield, see Diebold and Li (2006), the last row of A is defined as:

$$A_{(3,1:3),\tau_s} = \begin{bmatrix} 0 & 1 - \frac{1-exp(-\lambda_t \cdot \tau_s)}{\lambda_t \cdot \tau_s} & 1 - \left(\frac{1-exp(-\lambda_t \cdot \tau_s)}{\lambda_t \cdot \tau_s} - exp(-\lambda_t \cdot \tau_s)\right) \end{bmatrix}.$$

Specifying the curvature rotation as above ensures that the RDNS curvature loading converge to the DNS curvature loading for longer maturities.

The final issue is then to confirm that, $I = A^{-1} \cdot A$, i.e. that the square matrix A is non-singular. This can be done by showing that the determinant of A is different from zero.[3]

$$\begin{aligned} |A_{\tau_s}| &= a_{(1,1)} \cdot (a_{(2,2)} \cdot a_{(3,3)} - a_{(2,3)} \cdot a_{(3,2)}) \\ &\quad - a_{(1,2)} \cdot (a_{(2,1)} \cdot a_{(3,3)} - a_{(2,3)} \cdot a_{(3,1)}) \\ &\quad + a_{(1,3)} \cdot (a_{(2,1)} \cdot a_{(3,2)} - a_{(2,2)} \cdot a_{(3,1)}) \\ &= -exp(\lambda_t \cdot \tau_s). \end{aligned}$$

It is seen that for economically meaningful values of λ the determinant of the rotation matrix, $|A|$, is different from zero. For large values of λ the rotation matrix becomes singular, as $lim_{\lambda \to \text{inf}} |A| \to 0$. However, this is no surprise since also the Nelson-Siegel loading matrix, H, becomes somewhat meaningless in this case. In fact, the Nelson-Siegel model collapses to a single-factor model when λ approaches infinity.

As an illustration of the RDNS model, Figure (1) show the loading structures of the DNS and RNDS models, i.e. H and G respectively. The well-know loading pattern of the DNS model is seen in the first panel, where the Level, Slope, and Curvature loadings are recognized. In the

[3]The subscript τ_s is suppressed here for convenience.

second panel, the RDNS loadings are seen, and it is observed that they conform with the economic interpretation that was sought at the outset. One loading is impacting yields for all maturities in an identical way, with a loading of 1 for all maturities, just as the loading for the level factor in the DNS model. However, when seen in combination with the increasing concave function of the Slope loading, it is clear that the constant loading pattern of 1 in the RDNS corresponds to a Short Rate factor. The last hump-shaped loading represents the Curvature component of the model, as in the DNS model.

[** Insert Figure 1 around here **]

[** Insert Figure 2 around here **]

Figure 2 shows the time-series of the extracted yield curve factors from the DNS and RDNS models. When λ_t is fixed to a numerical value, e.g. 0.06, as opposed to being estimated, it is possible to solve for the factors using OLS. This is done here, as an illustration. In the empirical analysis of the RDNS model in following sections the factors are estimated using the kalman-filter.

As expected, the largest differences between panel 1, which shows the DNS factor time-series, and panel 2 that shows the RDNS factor time-series, are (a) that the first factor, which is shown by a continuous line, resembles the movements of the long end of the yield curve over the plotted data history - and thus has an interpretation as a Level factor, while the corresponding curve in panel 2 evolves according to the short end of the yield curve - and thus has an interpretation as a Short Rate factor; and (b) that the expected relationship $slope_{RDNS} = -slope_{DNS}$, so that a positive slope-factor observation in the RDNS model corresponds to an upward sloping yield curve.

3 The RDNS model with Macro Variables and VARMA dynamics

The basic principle of the RDNS model is presented in Section 2. In practical applications of the model, it may be relevant to include macro economic variables to help explain better the dynamic evolution of the yield curve factors over time, as in e.g. Diebold, Rudebusch, and Aruoba (2006), and to render model flexibility by allowing factors to evolve according to a vector autoregressive moving average process and by allowing the time-decay parameter, λ, to vary over time; see e.g. Koopmaman, Mallee, and der Wel (2010). To cater for these extensions I use a general state-space representation of the RDNS model that allows for the inclusion for two macro economic variables (industrial production and inflation), for VARMA dynamics, and for a time-varying time-decay parameter, λ_t.

In order to explore the relative importance of different model specifications, four concrete parametrizations are explored. These are:

1. a base-line parametrization with time-invariant parameters and no macro-economic variables;
2. a parametrization that allows for time-variation in the time-decay parameter, λ;
3. integrating inflation and industrial production as explanatory variables for the short rate factor and allowing for time-variation in the time-decay parameter, λ;
4. integrating macroeconomic variables and allowing for exogenously determined structural breaks in selected parameters of the model.

Model variant (1) represents a basic implementation of the RDNS model against which the other versions can be evaluated. This parametrization is similar to the DNS model first suggested by Diebold and Li (2006), although λ is estimated here, rather than being fixed to a certain value as in Diebold and Li (2006).[4] Specification (2) allows for time-variability in λ, and one estimate is produced for each date covered by the sample. Some authors find that the traditional approach to estimating the DNS model, where λ is fixed and not estimated, is sub-optimal. While fixing λ linearizes the DNS model, thus making it much easier to estimate, it is found empirically that λ exhibits a non-negligible degree of variation over time; see e.g. Koopmaman, Mallee, and der Wel (2010). The question is, of course, whether this time-variation affects the other model parameters, and the explanatory power of the model, in an economically and statistically significant way. Specification (3) integrates macroeconomic information. Theoretically, there are strong ties between the dynamic evolution of the macroeconomic environment and the yield curve; however empirically, in the context of DNS models as well as Gaussian arbitrage-free models, where macroeconomic information is integrated alongside lagged yield curve factors, the importance of macro variables is not found to be as strong as one may expect, see among others Diebold, Rudebusch, and Aruoba (2006). There are two (complementary) intuitive explanations for this empirical finding: (i) since three extracted yield curve factors typically explain more than 95% of the variability in a panel of yield curve data, and since the dynamics of these factors exhibit a high degree of persistence, there is very little information left to be explained by other non-yield-curve-factors, such as macroeconomic variables; (ii) linking the DNS factors to macroeconomic variables implies that the exogenous variables affect, in identical ways, the term premia and the underlying (risk free) yield curve dynamics. This may not necessarily be a valid assumption, for example

[4] Fixing λ is in general a good idea as it ensures that the model can be estimated by OLS.

because term permia are likely to be driven by the market's general level of risk aversion, while the risk free yield dynamics are closely related to the real growth rate of the economy and to inflation expectations. For this reason, it may be difficult to find a strong correlation between the DNS factors and macroeconomic variables. However, the RDNS model directly parameterizes the short rate as one of its yield curve factors, and the interpretation of, and separation between, the impact of macroeconomic variables on the short rate and on risk premia, are therefore less blurred.

Finally, specification (4) allows for structural breaks in a selected number of the estimated parameters. Such breaks are imposed exogenously, rather than estimated using an econometric technique, such as e.g. regime-switching. The break-dates represent 'macroeconomic event dates' that are often referred in the press and generally agreed upon by financial practitioners and academics. In particular, the period is split into three parts: the first period starts at the beginning of the data sample in January 1990 and covers the economic growth experienced during the 1990s until the burst of the dot com bubble in (around) November 2002. The second period spans the expansionary period from December 2002 to November 2008, when the sovereign and banking crisis had starting taking hold in Europe. And, the third period covers the remaining part of the data sample from December 2008 until June 2014, which represents a low growth and low inflationary economic environment.

One debatable issue is how macro economic variables are best included in factor-based term structure models. One view is that macro economic variables should be 'unspanned', and thus not affect the cross-sectional relationship between yields observed for different maturities at one given point in time, but that they may help explain risk premia and the dynamic evolution of yields over time. Another view is that traditional macro-finance models, where macro factors are spanned by the yield curve, work fine.[5] This discussion is of relevance to yield curve models that exclude arbitrage opportunities by construction, i.e. the class of multi-factor affine no-arbitrage models as introduced by Duffie and Kan (1996), and classified by Dai and Singleton (2000). The class of arbitrage free derived its factor-loading structure directly from the parameters that characterize the dynamic evolution of the yield curve factors, and a specification for the market price of risk. The loading structure therefore ensures consistency between the time-series dynamics of the model, and the model implied cross-sectional relationship between yields at each point in time: arbitrage opportunities are consequently excluded by construction. Since it is the $\mathbb{Q}$-measure parameters that feed into the calculation of the yield curve loading structure, it is necessary to impose restrictions on

[5]On this debate, see among others, Ludvigson and Ng (2009), Cooper and Priestly (2009), Duffee (2011), Joslin, Priebsch, and Singleton (2014), and Bauer and Rudebusch (2014b)

these parameters to ensure 'unspannedness' of macro variables, whereas the $\mathbb{P}$-measure dynamics can be left unconstrained. A Nelson-Siegel based model applied only under the observable $\mathbb{P}$ measure will, by definition, exclude any contemporaneous correlation between macro factors and the cross-sectional fit of yields (i.e. the model-implied yield curve) since its factor-loading structure derives from a pre-defined functional form that is unrelated to the parameters that characterize the dynamic evolution of the underlying yield curve factors. Dynamic Nelson-Siegel models will therefore, by construction, fulfill the 'unspannedness' requirement, unless the factor loading structure is altered accordingly.[6]

To integrate macroeconomic variables into the RDNS model, and to facilitate estimation of the four RDNS model variants mentioned above, the state equation (11) is expanded in the following way:

$$\gamma_t = m + F \cdot \gamma_{t-1} + Q \cdot M_t + W \cdot z_{t-1} + z_t, \tag{14}$$

where Q contains the coefficients accounting for the relationship between the yield curve factors and the macroeconomic variables $M = [CPI,\, IP]'$ (inflation and industrial production), and W contains the MA(1) coefficients.

The observation equation remains unchanged:

$$y_t(\tau) = G \cdot \gamma_t + e_t \tag{15}$$

3.1 The Term Structure of Term Premia in the RDNS model

The RDNS model facilitates direct calculation of fixed-income term premia, as the difference between the held-to-maturity return of an investment with maturity τ_i, and the strategy of rolling-over a short term investment for a time-period equal to τ_i. Denote by $rp_t(\tau_i)$ the term premium (risk premium) observed at time t for maturity τ_i, and, as above, τ_s, as the maturity of the short rate in the economy. Then, following e.g. Gürkaynak and Wright (2012), the general term premium expression is defined as:

$$rp_t(\tau_i) = y_t(\tau_i) - \frac{1}{\tau_i} \cdot E_t \left(\sum_{j=0}^{\tau_i - 1} y_{t+j}(\tau_s) \right).$$

[6] Versions of the Nelson-Siegel model can be derived to conform with no-arbitrage constraints, as shown by Christensen, Diebold, and Rudebusch (2011); in these models the discussion of 'unspanned-ness' of macro-factors may be relevant.

In the context of the RDNS model, the term structure of term premia can therefore be written as:[7]

$$rp_t(\tau_{i+1}) = G(\tau_{i+1}) \cdot \gamma_t - \mathbb{1} \cdot \frac{1}{\tau_{i+1}} \cdot \left[\sum_{j=1}^{\tau_i} j \cdot F^{\tau_i - j} \cdot (m + Q \cdot M_j) + \sum_{j=0}^{\tau_i} F^j \cdot \gamma_t + \sum_{j=0}^{\tau_i - 1} F^j \cdot W \cdot z_{t-1} \right], \quad (16)$$

where the first term on the RHS gives the yield for maturity τ_{i+1} based on the RDNS model shown in equation (4), and the second term is derived, using backward substitution, as the sum of the short rate factor, for period $t + \tau_i$. The dynamics of the short rate factor is given by the state equation of the RDNS model in (14), and $\mathbb{1} = [1, 0, 0]$ is a selection-vector for the short rate factor.

4 Data

The RDNS model is estimated using monthly US yield curve and macroeconomic data spanning the period from January 1990 to June 2014. In total there are 294 time-series observations for each of the included eleven maturities segments (i.e. $3, 12, 24, \ldots, 120$ months), and the two macroeconomic variables, inflation and industrial production. Summary statistics are shown in Table 1.

A familiar picture emerges of an on-average upward sloping yield curve, with a slope of approximately $200bp$, and with the standard deviation of yields that decreases through out the maturity spectrum. Also, as previously documented in the literature, yields exhibit strong serial autocorrelation.

Figure 3 display the time-series evolution of the the yields for the 3 months, 2 year and 10 year maturity segments in Panel 1, and Panel 2 shows the evolution of the macroeconomic series. The macroeconomic series are standardized to have a mean of zero and a unit variance.

[** Insert Table 1 around here **]

[** Insert Figure 3 around here **]

[7] For the practical implementation of the calculations, and to speed up calculation time, one can apply the closed form expression for the sum of a matrix power series, $F^0 + F^1 + F^2 + \cdots + F^{n-1} = (F^n - I)(F - I)^{-1}$, where I is the identity matrix.

5 Empirical Results

The main contribution of the current paper is to introduce the Rotated Dynamic Nelson-Siegel model, which is characterized by having underlying yield curve factors that have intuitive interpretations as the Short Rate, the Slope and the Curvature. Contrary to the economic interpretation of the DNS factors as Level, (minus) Slope, and Curvature, the factor structure of the RDNS model allows for integration of macroeconomic variables in a way similar to a Taylor-rule, i.e. where the exogenous variables directly impact the dynamic behavior of the short rate. In addition, the model facilitates computation of implied term premia.

The behavior of the model is investigated based on four different parameterizations:

1. a base-line parametrization with time-invariant parameters and no macro-economic variables;
2. allowing for time-variation in the time-decay parameter, λ;
3. integrating inflation and industrial production as explanatory variables for the short rate factor and allowing for time-variation in the time-decay parameter, λ;
4. integrating macroeconomic variables and allowing for exogenously determined structural breaks in selected parameters of the model.

The objective of the empirical analysis is not to search for the best performing model parametrization measured against some in-sample data-derived criteria, or to judge the model against its pseudo out-of-sample forecasting performance. As mentioned in the introduction, the forecasting performance of the baseline RDNS model will be identical to that of the DNS model - by construction - since the former is just a rotation of the latter. Naturally, once the RNDS model is augmented with e.g. macroeconomic variables, differences between the two model setups will emerge. In the context of the current paper, the main purpose of the empirical experiments is rather to illustrate to what extent the calculated term structure of term premia is sensitive to the chosen model parametrization.

Estimated parameters for the included model variants are shown in Tables 2 to 5. The commonly observed pattern of very persistent autoregressive yield curve factors re-emerges here: the diagonal elements of the F matrix in equation (14), i.e. $f_{1,1}, f_{2,2}, f_{3,3}$, are all close to, but less than, unity across all estimated characterizations. In addition, it appears that lagged values of the curvature factor help explain time-t observations of the Short Rate factor, as the estimate of $f_{1,3}$ is strongly statistically significant for all model specifications. Only the last time-period of model variant

4, covering the period from 2008 to 2014, deviates from the norm, with an effect that is only statistically significant at a 12% level. For all other parametrizations, the curvature factor affects the short rate with a statistical certainty of 99.9% or higher.

At first sight it appears relevant to include MA(1) terms in the dynamic evolution of the yield curve factors, as the MA(1) coefficients are different from zero at conventional levels of significance for most of the included model variants. Clearly, MA(1)-effects can be important for the in-sample fit of data, although their relevance in a forecasting context quickly disappears as the projection horizon increases. While MA(1) effects are statistically different from zero in the short rate equation across all four model specification, their effect on the slope and curvature factors depend on the over-all model specification. For example, in Model 2, which allows for a time-varying time-decay parameter λ, the MA(1) coefficients in the slope and curvature equations are no-longer estimated to be significantly different from zero at conventional levels of significance.

Models 3 and 4 include macro economic variables in their description of the short rate process. It is a well-established fact in the empirical yield curve literature, both for affine models and the Nelson-Siegel type models, that the persistence of the yield curve factors leave little room for explanatory power to be derived from including exogenous variables. This result is replicated in model variant 3, which documents that there is no statistically significant impact from industrial production or inflation on the evolution of the short rate, once autoregressive effects from the short rate, and the influence the lagged curvature factor, are included. However, once the data set is split into sub-periods, as in Model 4, the macro economic variables contribute significantly to the dynamic evolution of the yield curve factors between 1990 to 2002, while they display no significant contribution in the last two sub-periods. Based purely on the way that the yield curve data evolves, the distinct inverse U-shaped pattern that the short rate traces during the period from 2002 to 2008 can clearly be well explained by an autoregressive process alone: the first part of the period exhibits a strong positive 'trend', while the latter part, from 2007 to 2008, shows a strong negative trend. As such, this particular path followed by the short rate (and the rest of the yield curve maturities) during this period leaves no room for other explanatory variables, besides the already included VAR-elements. In the last sub-period, the yield curve evolution is equally extreme, although in a different way. During this period, the short rate shows very little variation and its level is very close to zero. Clearly, this type of time-series behavior does not leave room for explanatory power to be provided by exogenous macroeconomic variables.[8]

[8]This conclusion may naturally be reversed, if the model was specified in terms of a shadow short rate, which would not be constrained to be strictly positive, e.g. following Black (1995), Christensen and Rudebusch (2013), Kim and Priebsch (2013), Bauer and Rudebusch (2014a), Krippner (2013), Kim and Singleton (2012), and Wu and

A time-varying time-decay parameter, λ, is included in model variants 2 and 3. Model 4 includes a λ-parameter that can vary across the three pre-defined time-periods. As observed in Figures 4 and 5, the λ-parameter for models 2 and 3 shows a somewhat stable pattern, where λ for most dates falls between 0.025 and 0.058.[9] Occasionally, this stable pattern is interrupted by single spikes. Also for model 4 little variation is observed in the λ parameters in each sub-period. Indeed, when looking at the average values for the time-decay parameter in models 2 and 3, in combination with the λs of model 4, it is difficult to justify that the λ-parameter should be allowed to vary over time; especially when considering that the Short Rate, Slope, and Curvature factors easily can adapt such that a constant λ will have no adverse effects on the fit of the model. While no statistical test is performed to support this observation, it is noted that BIC is -4534 for model specification 1 and -4532 for both specification 2 and 3. These figures imply that there is only a marginal difference between the fits of a model that allow for time-varying λ and to the alternative model where λ is constant.

[** Insert Table 2 around here **]

[** Insert Table 3 around here **]

[** Insert Table 4 around here **]

[** Insert Table 5 around here **]

[** Insert Figure 4 around here **]

[** Insert Figure 5 around here **]

Tables 6 to 9 report summary statistics for the estimated term structure of term permia, and Figures 6 to 9 provide visual representations of the time-series evolution of the term premia at selected horizons of 6 months, 1, 5, and 10 years. The figures also show the path followed by the short rate on each date covered by the sample; discrete sampling points on these paths for horizons of 6 months, 1, 5, and 10 years are included. As such, the figures give a view on the left-hand-side of the term-premia expression in equation (16), i.e the estimated term premia, and on the second term on the right-hand-side of the same equation, i.e. the model implied yield curve derived from the dynamics of the short rate process.

Xia (2014). A shadow short rate version of the RDNS model can be derived.

[9]The referred λ-values correspond to the 25% and 75% quantiles of the distributions for λ in models 2 and 3, i.e. these quantiles are identical across the these to model specifications.

According to Tables 6, 7, and 8, there is little practical difference between the term premia estimates of model variants 1 to 3. Visual inspection of the respective Figures 6 to 8 confirms this observation. Consequently, for the estimation of implied term premia, it makes little difference whether λ is treated as fixed or time-varying, and whether MA(1) effects are included in the model; clearly, macroeconomic variables will make little difference if they are included as in model variant 3, where they, as mentioned above, fail to attract statistically significant parameter estimates.

On this basis, it can be concluded that model variants 1 to 3 provide indistinguishable term-premia estimates. Model variant 4 is, however, different. As documented in Table 9. Furthermore, model variant 4 provides a more clear picture on the term-premium compression in the short end of the maturity spectrum, starting around year 2008, while model variants 1 to 3 draw a picture of a somewhat continuous premia compression throughout the whole analyzed period, i.e. from 1990 to 2014. For longer maturities, i.e. 5years and 10years, all model variants agree that term premia have been on a declining trend since 1990.

Another difference between the tested model specifications is worth highlighting. As mentioned above, the upper panel of Figures 6 to 9 shows the model implied path for the policy rate at each date covered by the sample. Model variant 4 implies decreasing policy rates consistently from 1990 to 2014, since the policy rate at the 10year horizon is lower than the 6 months maturity for all sampling points. And, not only is the policy rate expected to fall, it is expected to fall to levels that are far below what any reasonable estimate for the real growth rate would predict.[10] In fact, the upper panel of Figure 9 shows that this model variant predicts that the long-end of the risk-free term structure, i.e. the average cumulative policy rate over the 10 years horizon, is less than unity for all dates in the sample. Such a policy rate development is not realistic. Contrary to this, model variants 1 to 3 produce reasonable policy rate projections: the policy rate is assumed to fall during the period from 1990 to 2002, with its attraction point varying between 3% and 4%. From 2002 to 2005 the policy rate path is constant across the included horizons, and its attraction point is roughly 1.5% which implies that investors during this period believed that the growth rate of the economy would be somewhat below historical averages. The period from 2005 to 2008 resembles the model implied policy rate expectation during the period from 1990 to 2002. And, then after 2008 the policy rate expectation dropped to around 1%, at the 10 year horizon, and to 0% at the 6 months horizon.[11]

[10] A standard economic equilibrium argument would suggest that the long end of the risk-free government yield curve should equal the real growth rate of the economy, since the yield is the 'price of financing', which, in a risk-free world that is in equilibrium, would equal the value-added of real investment, which by definition is the growth rate of the economy in question.

[11] It should be noted that the estimated factor dynamics for all models indicate that the yield curve data series

When comparing and contrasting the economic interpretation of the tested model specifications, it is clear that variants 1 to 3 produce sensible results that are in line with the observed history, while variant 4 does not.

Lastly it is relevant to compare the estimated term premia from the RDNS model with estimates that already exist in the literature. Two models act as natural benchmarks in this regard, namely Adrian, Crump, and Mönch (2013) (ACM) and Kim and Wright (2005) (KW). Figure 10 displays the time series evolution of the 10year term premiums from the ACM, KW and RDNS models, from January 1990 to June 2014. Equation 16 shows that the unconditional mean of the short rate process determines the level of the estimated term premia, and it may therefore be less relevant whether the actual levels of historical term premia, observed for similar maturities, compare well with each other over time, as the level of the short rate will be highly dependent on the data window used to estimate the model. Rather, it is relevant to compare the dynamic evolution of the premia. To illustrate that the level of estimated term premia depend on the attraction point for the estimated short rate process, the RDNS model that is used to produce Figure 10 constrains the mean of the first model-factor (i.e. the short rate) to equal the historical mean of the maturity matched yield curve segment. Consequently, the mean of the short rate factor is, by construction, set equal to the historical average of the observed three month yield.[12]

Inspection of Figure 10 shows that the 10year RDNS term premium estimate is very similar, both in terms of level and dynamic evolution, to the 10 year premia derived from the ACM and KW models. To complement this visual impression, it is noted that the correlation between the RDNS and ACM estimates is 0.95 (0.92 when the correlation is calculated between the first difference of the series), and the correlation between the RDNS and KW estimates is 0.90 (0.83 in first differences), while the correlation between the ACM and KW estimates is 0.77 (0.72 in first differences).

[** Insert Table 6 around here **]

[** Insert Table 7 around here **]

[** Insert Table 8 around here **]

[** Insert Table 9 around here **]

[** Insert Figure 6 around here **]

are close to being integrated of order one. The autoregressive parameters shown in Tables 2 to 5, for each estimated equation, is less, but close to 1. A direct economic translation of the presented results should therefore be done with caution.

[12] Note that this constraint is not imposed on Models 1 to 4 estimated in the paper.

[** Insert Figure 7 around here **]

[** Insert Figure 8 around here **]

[** Insert Figure 9 around here **]

[** Insert Table 10 around here **]

6 Conclusion

I introduce a new parametrization of the well-known Dynamic Nelson-Siegel yield curve model-family where the underlying factors have interpretations as the short rate, the slope, and the curvature. This contrasts with the traditional Nelson-Siegel model where factors can be interpreted as the level, (minus) the slope and the curvature. The proposed Rotated Dynamic Nelson-Siegel (RDNS) facilitates easy calculation of the term-structure of term premia and integration of macroeconomic variables into the model via a Taylor-rule type equation.

The RDNS model is estimated on US data observed monthly and covering the period from 1990 to 2014. Four empirical experiments are performed to illustrate how the model works. I examine whether it adds value to include a time-varying time-decay parameter, whether macroeconomic variables are needed to explain the time-series dynamics of the short rate and term premia, and whether is it useful to impose structural breaks according to historically observed business cycle dates.

It is found that term premia generated by the model are economically meaningful, although care needs to be taken with respect to how the model is parameterized. There is little added value from allowing the time-decay parameter to vary over the sample, and macroeconomic variables are only statistically significant for the data period from 1990 to 2002.

Given the constraints imposed on the short rate process by the zero-lower bound, it is perhaps not so surprising that macroeconomic variables play little role for the model variants included in the current paper. It is likely that a clear Taylor-type relationship will emerge if a shadow short-rate, i.e. a process that is not constrained by the zero-lower bound, is included in the RDNS model. It is left for future research to investigate this question.

It is also relevant to investigate how an arbitrage-free version of the RDNS model, i.e. a model akin to a rotated version of Christensen, Diebold, and Rudebusch (2011), performs in comparison to the RDNS model presented above. This topic is also left for future research.

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Tables and Figures

Maturity	Mean	Std	Min	Max	$\rho(1)$	$\rho(2)$	$\rho(3)$	$\rho(6)$
3M	3.164	2.305	0.005	8.049	0.985	0.967	0.946	0.874
1Y	3.480	2.386	0.098	8.730	0.984	0.965	0.943	0.870
2Y	3.814	2.392	0.204	9.179	0.984	0.963	0.941	0.874
3Y	4.046	2.311	0.289	9.190	0.983	0.962	0.940	0.875
4Y	4.285	2.223	0.432	9.206	0.982	0.960	0.938	0.874
5Y	4.506	2.152	0.593	9.224	0.981	0.959	0.938	0.874
6Y	4.672	2.065	0.804	9.226	0.980	0.957	0.936	0.870
7Y	4.831	1.975	0.998	9.230	0.979	0.955	0.933	0.867
8Y	4.958	1.916	1.150	9.234	0.979	0.955	0.934	0.867
9Y	5.071	1.858	1.360	9.239	0.979	0.954	0.933	0.868
10Y	5.151	1.832	1.553	9.244	0.978	0.954	0.934	0.870
CPI	0.000	1.000	-1.977	3.234	0.983	0.962	0.940	0.864
IP	0.000	1.000	-4.176	1.596	0.974	0.938	0.889	0.670

The table shows summary statistics for the included data series. Time series of yield observed from 3 months to 10 years and economic growth (Industrial Production) and the inflation rate (Consumer Price Index) are used in the model and means, the standard deviation, the minimums and maximums are reported. In addition, $\rho(p)$ reports the autocorrelation for each series at lag p. Note that the macroeconomic variables have been standardized by subtracting the empirical mean and divining each observation with the estimated standard deviation.

Table 1: Summary Statistics of the Data

Parameter	Estimate	Stderr	t-value	p-value
		Time-decay parameter		
λ_1	0.039	0.002	25.399	0.000
		Constants		
k_1	0.183	0.135	1.354	0.176
k_2	0.275	0.286	0.960	0.337
k_3	-0.660	0.326	-2.023	0.043
		AR coefficients		
$f_{1,1}$	0.957	0.025	37.939	0.000
$f_{2,1}$	-0.029	0.052	-0.556	0.578
$f_{3,1}$	0.112	0.054	2.061	0.039
$f_{1,2}$	0.000	0.015	0.000	1.000
$f_{2,2}$	0.942	0.035	26.852	0.000
$f_{3,2}$	0.000	0.046	0.000	1.000
$f_{1,3}$	0.040	0.014	2.879	0.004
$f_{2,3}$	0.000	0.026	0.000	1.000
$f_{3,3}$	0.842	0.054	15.691	0.000
		MA coefficients		
ma_1	0.108	0.013	8.516	0.000
ma_2	0.311	0.022	14.083	0.000
ma_3	0.245	0.116	2.109	0.035
		Macro coefficients		
f_{CPI}	na	-	-	-
f_{IP}	na	-	-	-

The table shows parameter estimates obtained from the Rotated Dynamic Nelson-Siegel (RDNS) model following specification 1 on page 8: the base-line specification with time-invariant parameters and no macroeconomic variables.

Table 2: Parameter Estimates - Model 1

Parameter	Estimate	Stderr	t-value	p-value
		Time-decay parameter		
$\lambda_{average}$	0.056	0.010	18.392	0.008
		Constants		
k_1	0.151	0.042	3.601	0.000
k_2	-0.030	0.058	-0.520	0.603
k_3	-0.282	0.168	-1.681	0.093
		AR coefficients		
$f_{1,1}$	0.966	0.007	132.364	0.000
$f_{2,1}$	0.013	0.011	1.131	0.258
$f_{3,1}$	0.036	0.028	1.301	0.193
$f_{1,2}$	0.000	0.013	0.000	1.000
$f_{2,2}$	0.999	0.006	176.114	0.000
$f_{3,2}$	0.000	0.024	0.000	1.000
$f_{1,3}$	0.028	0.010	2.949	0.003
$f_{2,3}$	0.000	0.008	0.000	1.000
$f_{3,3}$	0.937	0.016	57.089	0.000
		MA coefficients		
ma_1	0.037	0.004	8.778	0.000
ma_2	-0.016	0.012	-1.303	0.192
ma_3	0.030	0.030	1.001	0.317
		Macro coefficients		
f_{CPI}	na	-	-	-
f_{IP}	na	-	-	-

The table shows parameter estimates obtained from the Rotated Dynamic Nelson-Siegel (RDNS) model following specification 2 on page 8: with a time-varying time-decay parameter, λ_t and no macroeconomic variables.

Table 3: Parameter Estimates - Model 2

Parameter	Estimate	Stderr	t-value	p-value
		Time-decay parameter		
$\lambda_{average}$	0.056	0.065	21.919	0.007
		Constants		
k_1	0.152	0.028	5.359	0.000
k_2	-0.028	0.033	-0.861	0.389
k_3	-0.281	0.088	-3.188	0.001
		AR coefficients		
$f_{1,1}$	0.966	0.005	178.828	0.000
$f_{2,1}$	0.012	0.007	1.700	0.089
$f_{3,1}$	0.036	0.020	1.833	0.067
$f_{1,2}$	0.000	0.000	0.000	0.000
$f_{2,2}$	0.999	0.003	324.915	0.000
$f_{3,2}$	0.000	0.000	0.000	0.000
$f_{1,3}$	0.028	0.005	6.095	0.000
$f_{2,3}$	0.000	0.000	0.000	0.000
$f_{3,3}$	0.937	0.010	90.950	0.000
		MA coefficients		
ma_1	0.037	0.003	10.673	0.000
ma_2	-0.016	0.007	-2.341	0.019
ma_3	0.030	0.026	1.141	0.254
		Macro coefficients		
f_{CPI}	0.002	0.005	0.355	0.723
f_{IP}	0.001	0.004	0.190	0.849

The table shows parameter estimates obtained from the Rotated Dynamic Nelson-Siegel (RDNS) model following specification 3 on page 8: with a time-varying time-decay parameter, λ_t and macroeconomic variables.

Table 4: Parameter Estimates - Model 3

Parameter	Estimate	Stderr	t-value	p-value
	Time-decay parameter - 1990 to 2002			
λ_1	0.047	0.005	9.786	0.000
	Time-decay parameter - 2002 to 2008			
λ_2	0.041	0.003	14.295	0.000
	Time-decay parameter - 2008 to 2014			
λ_3	0.037	0.002	21.796	0.000
	Constants - 1990 to 2002			
k_1	0.250	0.307	0.816	0.414
k_2	0.336	0.534	0.630	0.529
k_3	-0.442	0.497	-0.889	0.374
	Constants - 2002 to 2008			
k_1	0.371	0.371	1.000	0.317
k_2	1.961	1.005	1.952	0.051
k_3	-1.459	2.431	-0.600	0.549
	Constants - 2008 to 2014			
k_1	0.294	0.227	1.296	0.195
k_2	0.950	0.767	1.239	0.215
k_3	-1.298	0.847	-1.533	0.125

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Parameter	Estimate	Stderr	t-value	p-value
AR coefficients - 1990 to 2002				
$f_{1,1}$	0.930	0.055	16.891	0.000
$f_{2,1}$	-0.038	0.093	-0.404	0.686
$f_{3,1}$	0.067	0.075	0.894	0.371
$f_{1,2}$	0.000	0.043	0.000	1.000
$f_{2,2}$	0.941	0.073	12.843	0.000
$f_{3,2}$	0.000	0.065	0.000	1.000
$f_{1,3}$	0.060	0.023	2.627	0.009
$f_{2,3}$	0.000	0.034	0.000	1.000
$f_{3,3}$	0.743	0.086	8.688	0.000
AR coefficients - 2002 to 2008				
$f_{1,1}$	0.939	0.075	12.521	0.000
$f_{2,1}$	-0.387	0.204	-1.895	0.058
$f_{3,1}$	0.166	0.481	0.345	0.730
$f_{1,2}$	0.000	0.062	0.000	1.000
$f_{2,2}$	0.662	0.164	4.030	0.000
$f_{3,2}$	0.000	0.401	0.000	1.000
$f_{1,3}$	0.073	0.027	2.718	0.007
$f_{2,3}$	0.000	0.057	0.000	1.000
$f_{3,3}$	0.607	0.150	4.037	0.000
AR coefficients - 2008 to 2014				
$f_{1,1}$	0.910	0.189	4.823	0.000
$f_{2,1}$	0.000	0.924	0.000	1.000
$f_{3,1}$	0.000	1.106	0.000	1.000
$f_{1,2}$	0.000	0.036	0.000	1.000
$f_{2,2}$	0.804	0.135	5.976	0.000
$f_{3,2}$	0.000	0.144	0.000	1.000
$f_{1,3}$	0.067	0.043	1.550	0.121
$f_{2,3}$	0.000	0.118	0.000	1.000
$f_{3,3}$	0.749	0.144	5.207	0.000

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Parameter	Estimate	Stderr	t-value	p-value
		MA coefficients		
ma_1	0.091	0.010	9.147	0.000
ma_2	0.316	0.028	11.412	0.000
ma_3	0.285	0.115	2.486	0.013
		Macro coefficients - 1990 to 2002		
f_{CPI}	0.071	0.068	1.048	0.294
f_{IP}	0.129	0.040	3.243	0.001
		Macro coefficients - 2002 to 2008		
f_{CPI}	0.010	0.057	0.167	0.867
f_{IP}	-0.003	0.069	-0.038	0.970
		Macro coefficients - 2008 to 2014		
f_{CPI}	0.000	1.106	0.000	1.000
f_{IP}	0.000	0.036	0.000	1.000

The table shows parameter estimates obtained from the Rotated Dynamic Nelson-Siegel (RDNS) model following specification 4 on page 8: with a time-varying time-decay parameter, λ_t, macroeconomic variables, and structural breaks in selected model parameters.

Table 5: Parameter Estimates - Model 4

Maturity	Mean	Std	Min	Max	$\rho(1)$	$\rho(2)$	$\rho(3)$	$\rho(6)$
6M	0.114	0.095	-0.081	0.355	0.804	0.814	0.766	0.709
1Y	0.338	0.239	-0.195	0.931	0.930	0.909	0.890	0.811
5Y	1.816	0.827	0.102	3.908	0.965	0.928	0.905	0.802
10Y	2.863	1.013	0.783	5.336	0.963	0.926	0.907	0.813

The table shows summary statistics for the Term Premia estimates obtained from the Rotated Dynamic Nelson-Siegel (RDNS) model following specification 1 on page 8: the base-line specification with time-invariant parameters and no macroeconomic variables.

Table 6: Term Premia Summary Statistics - Model 1

Maturity	Mean	Std	Min	Max	$\rho(1)$	$\rho(2)$	$\rho(3)$	$\rho(6)$
6M	0.117	0.140	-0.099	1.217	0.813	0.669	0.590	0.473
1Y	0.340	0.325	-0.169	2.046	0.902	0.803	0.736	0.591
5Y	1.801	0.929	0.254	4.289	0.944	0.886	0.847	0.720
10Y	2.820	1.085	0.894	5.598	0.956	0.910	0.883	0.780

The table shows summary statistics for the Term Premia estimates obtained from the Rotated Dynamic Nelson-Siegel (RDNS) model following specification 2 on page 8: with a time-varying time-decay parameter, λ_t and no macroeconomic variables

Table 7: Term Premia Summary Statistics - Model 2

Maturity	Mean	Std	Min	Max	$\rho(1)$	$\rho(2)$	$\rho(3)$	$\rho(6)$
6M	0.117	0.140	-0.101	1.215	0.811	0.666	0.587	0.469
1Y	0.338	0.324	-0.175	2.043	0.901	0.802	0.735	0.589
5Y	1.796	0.929	0.254	4.285	0.944	0.886	0.846	0.720
10Y	2.811	1.085	0.892	5.589	0.956	0.910	0.883	0.780

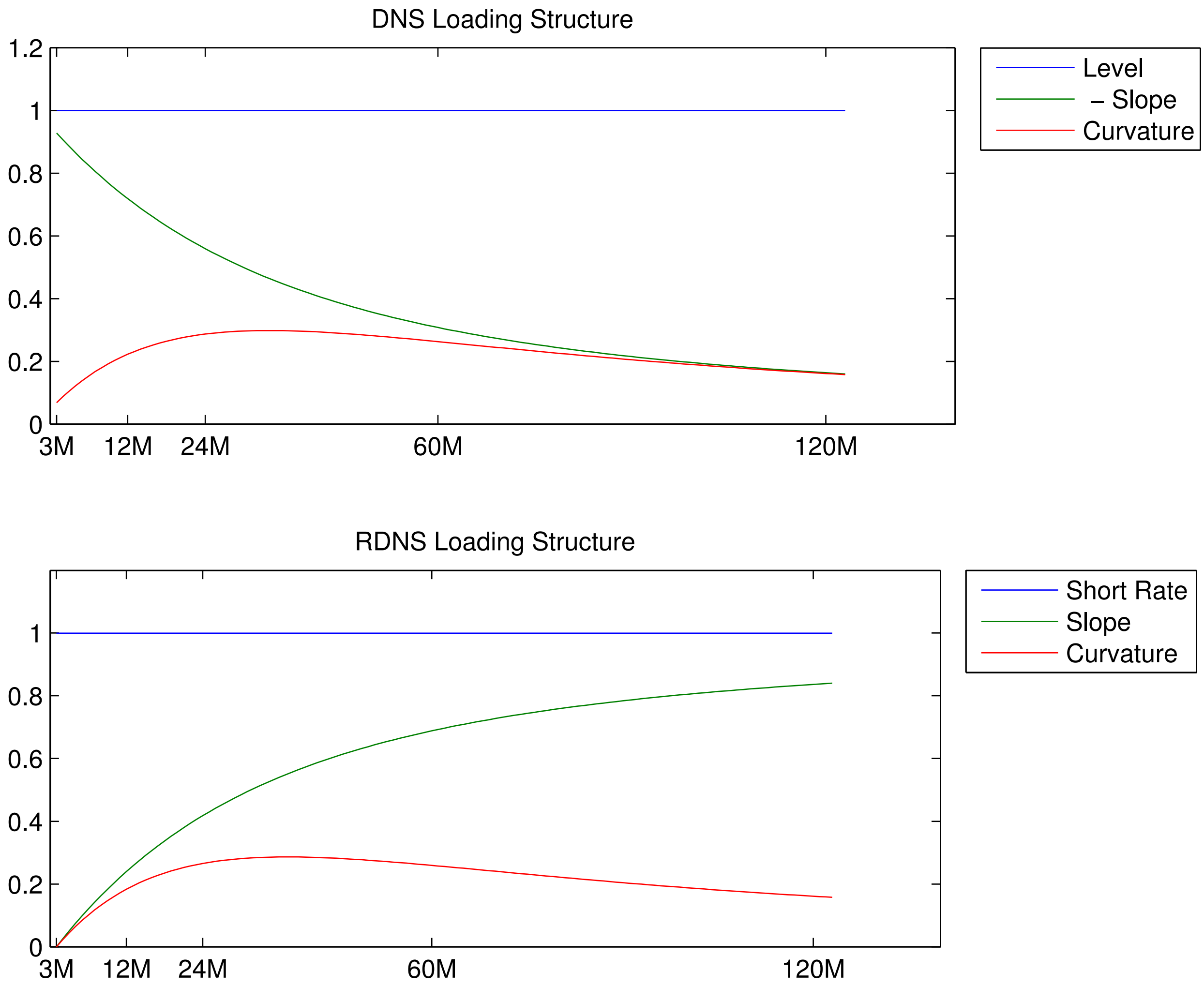
The table shows summary statistics for the Term Premia estimates obtained from the Rotated Dynamic Nelson-Siegel (RDNS) model following specification 3 on page 8: with a time-varying time-decay parameter, λ_t and macroeconomic variables

Table 8: Term Premia Summary Statistics - Model 3

Maturity	Mean	Std	Min	Max	$\rho(1)$	$\rho(2)$	$\rho(3)$	$\rho(6)$
6M	0.302	0.214	-0.043	0.939	0.949	0.923	0.891	0.815
1Y	1.035	0.579	0.012	2.470	0.978	0.957	0.936	0.867
5Y	4.110	1.607	1.021	7.817	0.979	0.953	0.929	0.859
10Y	5.252	1.558	1.996	8.854	0.975	0.946	0.923	0.850

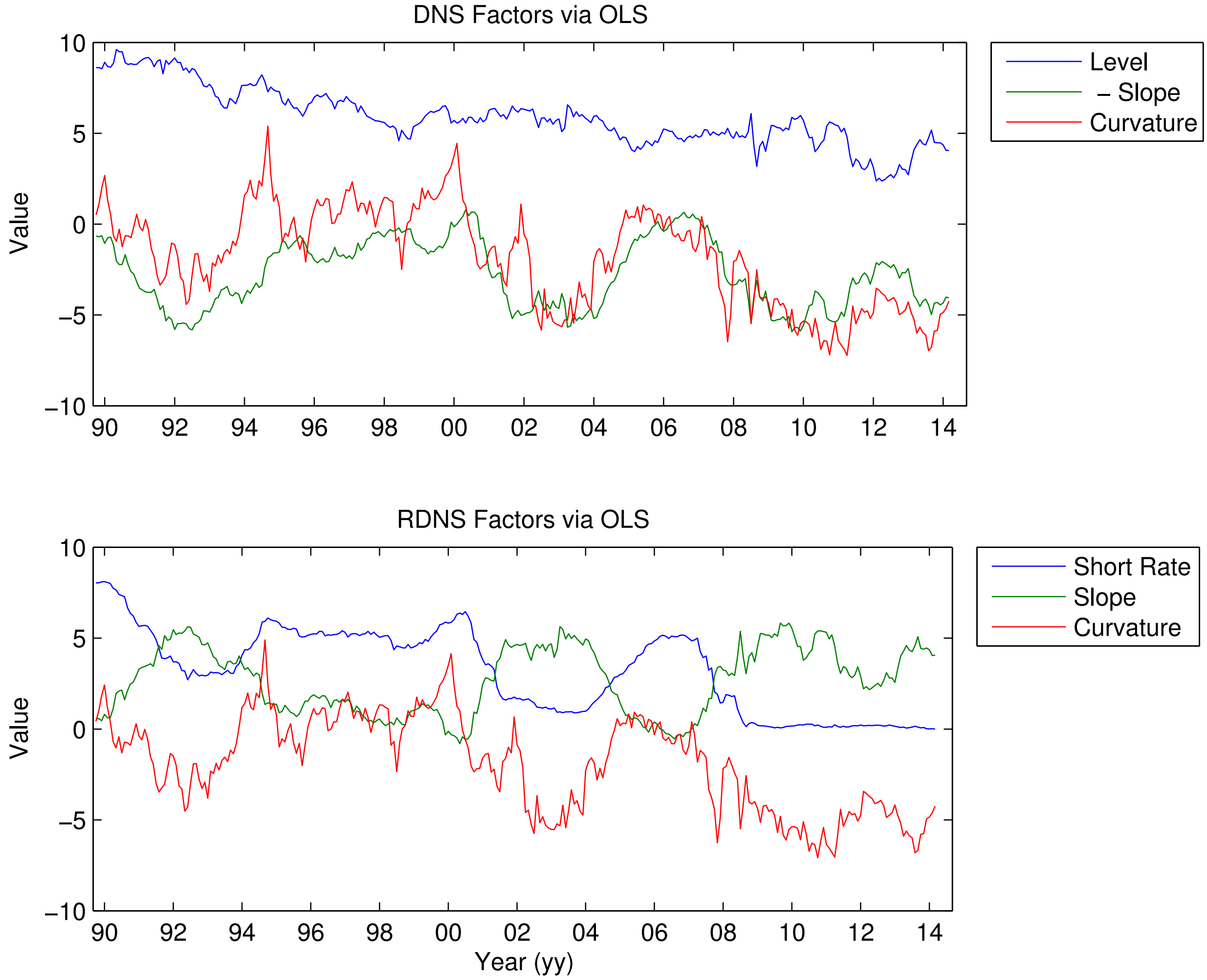
The table shows summary statistics for the Term Premia estimates obtained from the Rotated Dynamic Nelson-Siegel (RDNS) model following specification 4 on page 8: with a time-varying time-decay parameter, λ_t, and macroeconomic variables, as well as structural breaks in selected model parameters.

Table 9: Term Premia Summary Statistics - Model 4



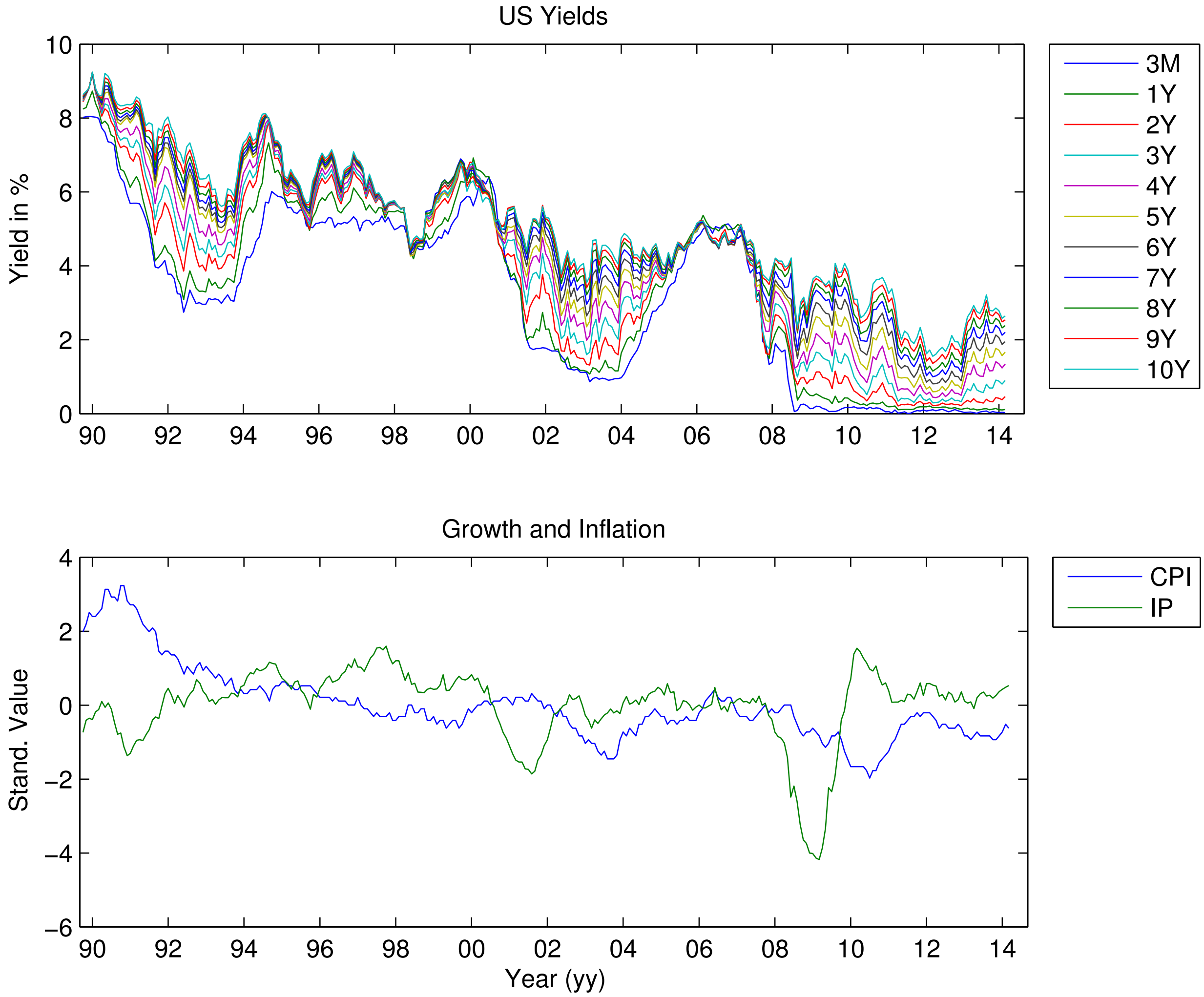
The upper panel of this figure shows the loading pattern of a traditional Nelson-Siegel based model. The lower panel shown the loading pattern of the Rotated Nelson-Siegel model.

Figure 1: Loading Structures



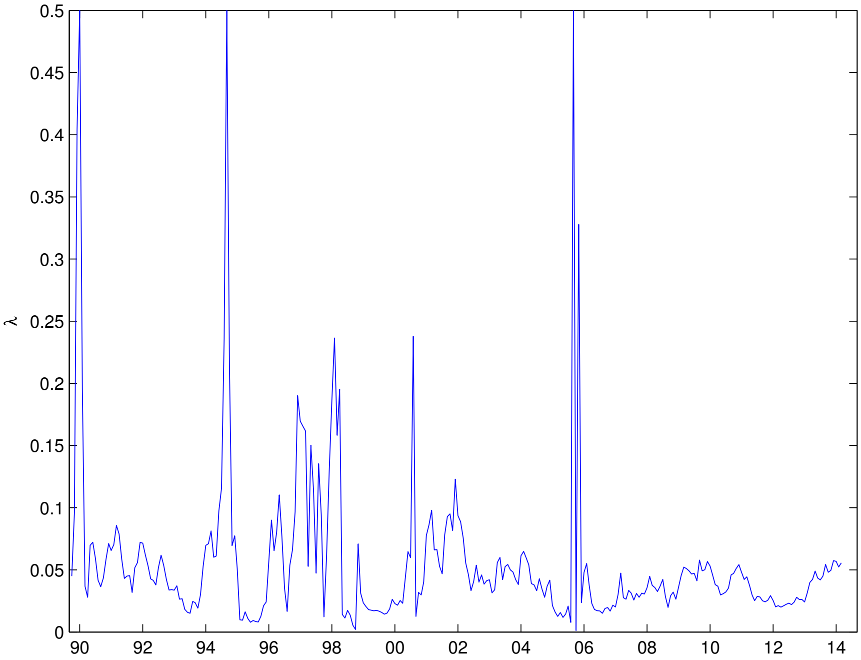
The upper panel shows the dynamic evolution of the three traditional Nelson-Siegel factors: Level, minus Slope, and curvature. The lower panel shows the dynamic evolution of the factors obtained from the Rotated Nelson-Siegel model, corresponding to: the Short Rate, the Slope, and the Curvature. In both panel Ordinary Least Squares has been used to extract the factors, i.e it is assumed that the time-decay parameter is known and that no autoregressive effects are accounted for in the dynamic evolution of the factors. Hence, the factors can be found as: $\beta_{NS}^{OLS} = (H^T \cdot H)^{-1} \cdot H^T \cdot Y$, $\beta_{RNS}^{OLS} = (G^T \cdot G)^{-1} \cdot G^T \cdot Y$, where Y is the panel of observed yields and H and G are the loading matrices in equation (1) and equation (10), respectively.

Figure 2: Yield Curve Factors



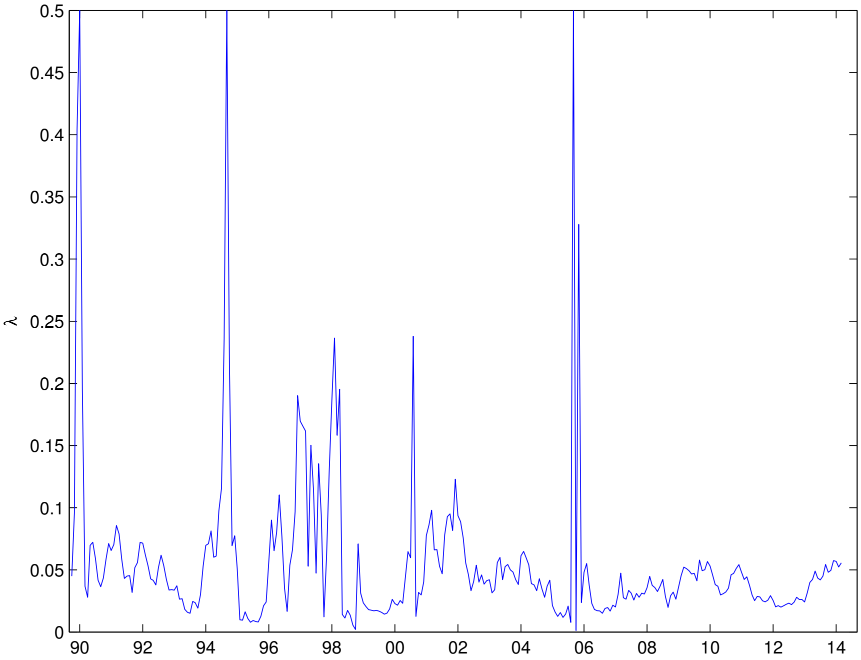
This Figure shows in the upper panel the used yield curve data spanning the period from 1990 to 2014 and observed for maturities at maturities covering 3 months to 10 years. The lower panel shows the included macroeconomic variables, industrial production (IP) and the inflation rate (CPI). The macro series have been standardized by subtracting the empirical mean and dividing each observation by the empirical standard deviation.

Figure 3: US Data



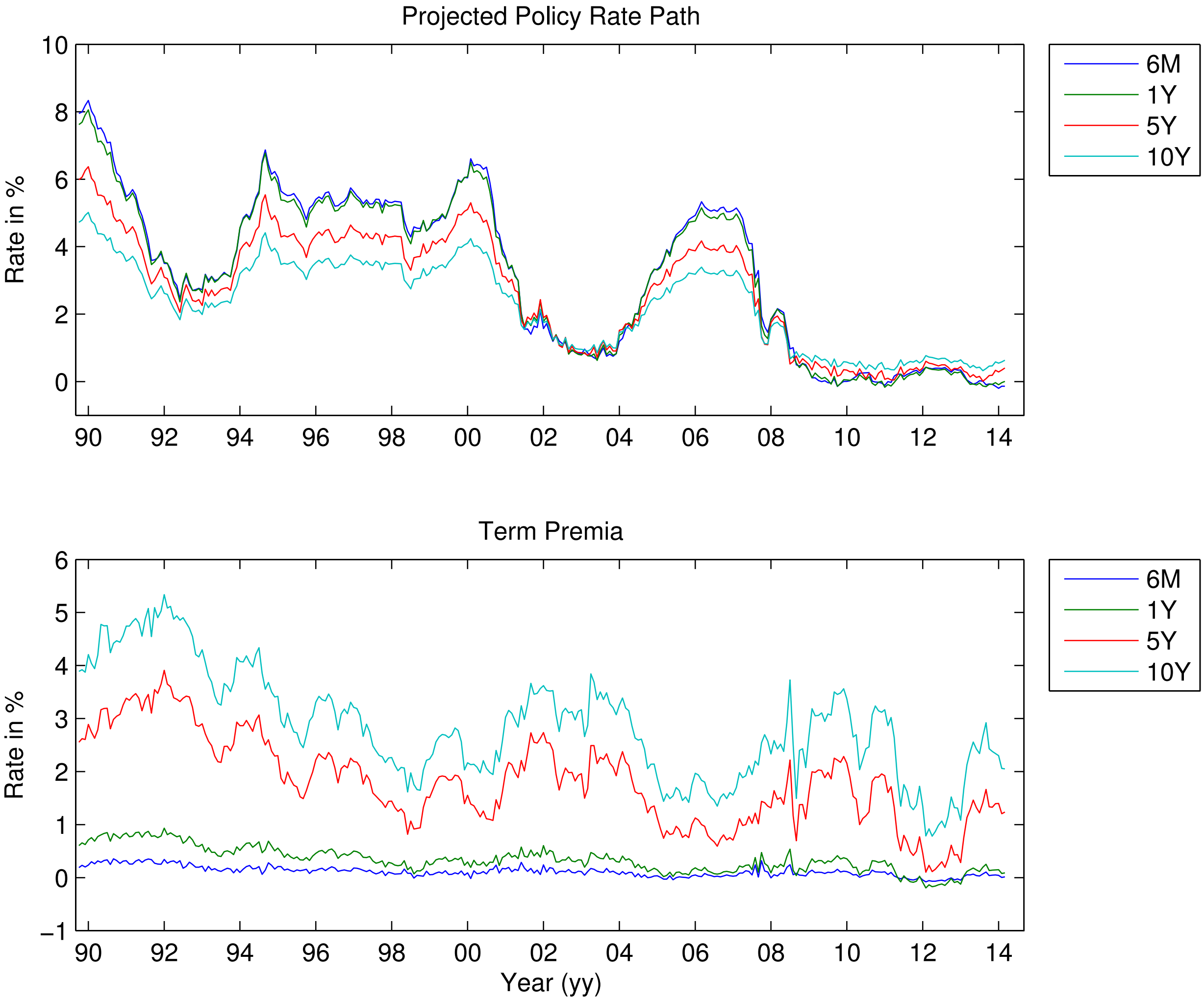
This Figure shown the time-series evolution of λ_t obtained from from the Rotated Dynamic Nelson-Siegel (RDNS) model following specification 2 on page 8: with a time-varying time-decay parameter, λ_t and no macroeconomic variables.

Figure 4: Time Varying λ Parameter - Model 2



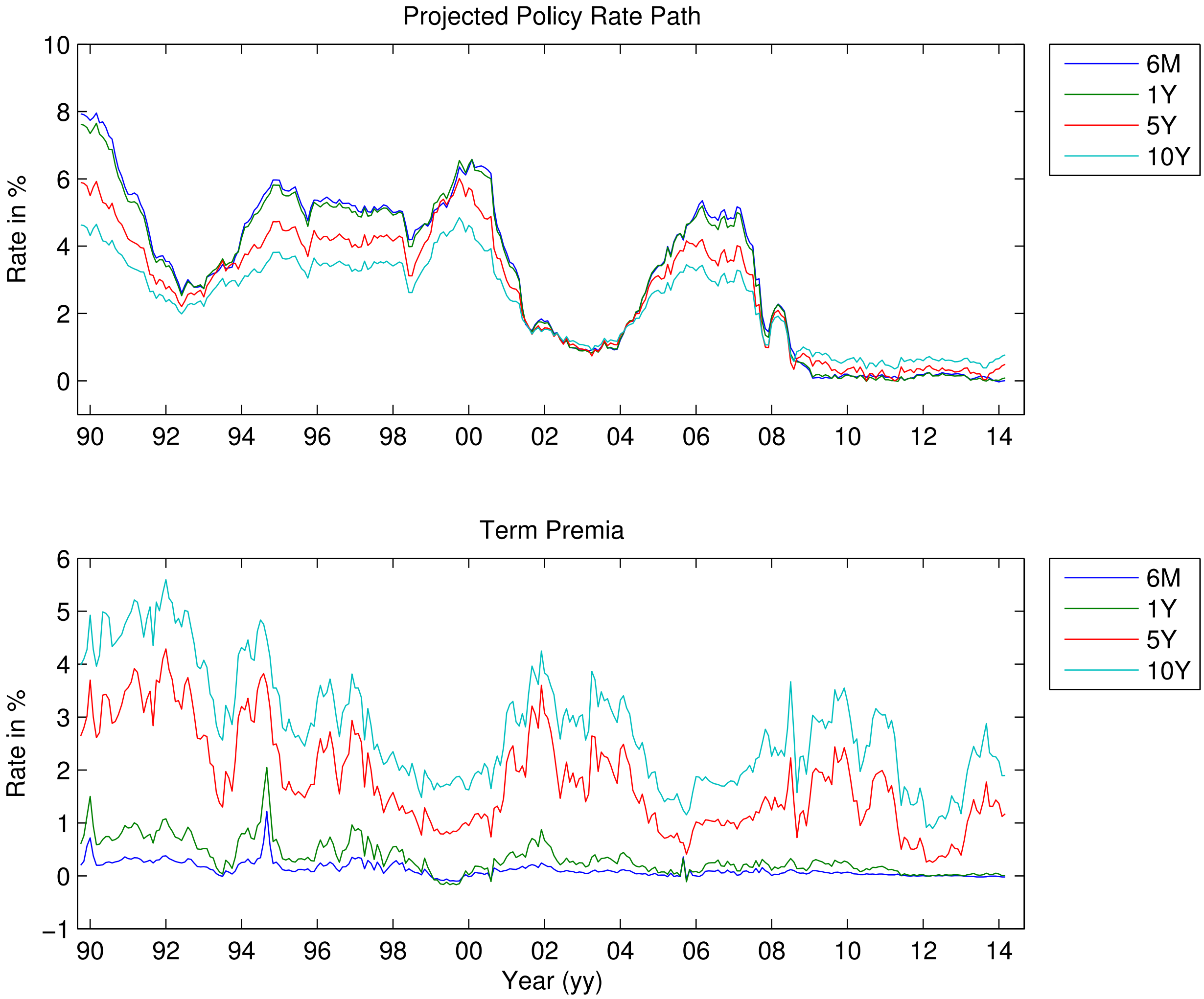
This Figure shown the time-series evolution of λ_t obtained from from the Rotated Dynamic Nelson-Siegel (RDNS) model following specification 3 on page 8: with a time-varying time-decay parameter, λ_t and no macroeconomic variables.

Figure 5: Time Varying λ Parameter - Model 3



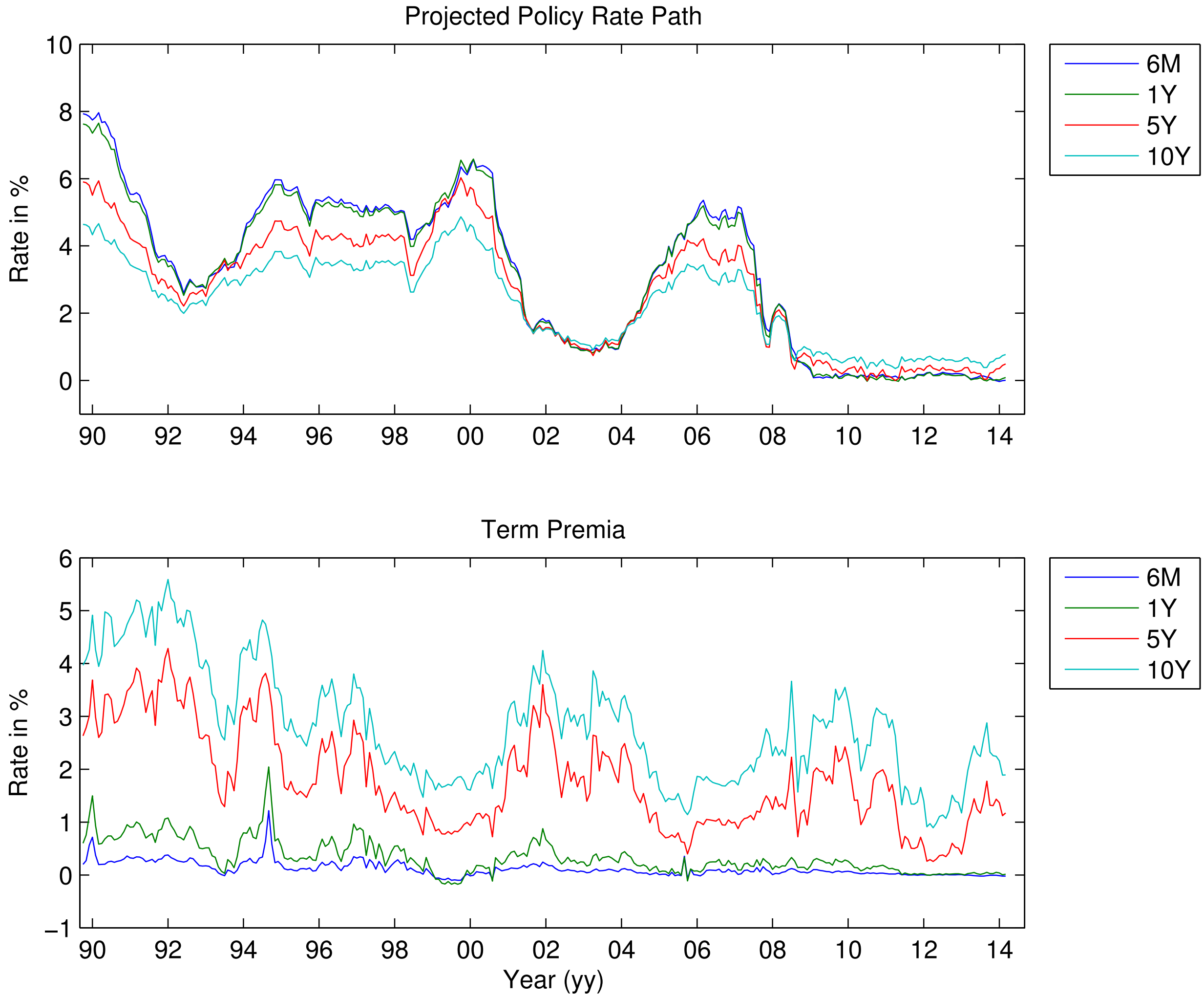
The upper panel shows the evolution over time of the projected path for the policy rate at the 6 months, 1 year, 5 year and 10 years projection horizons. At a given point in time, say in 2006 (06 on the x-axis of the panel) the lines show the expected location of the policy rate over the respective projection horizon. In other words, this panel depicts the accumulation of the short rate, using the model parameters, for the selected horizons of 6 months to 10 years. The lower panel shows the term premia for the 6 months, 1 year, 5 year and 10 years horizons, as defined by equation (16), which effectively is the difference between the model derived yield curve and the model projections made for the short rate. Results shown in the figure are derived from from the Rotated Dynamic Nelson-Siegel (RDNS) model following specification 1 on page 8: the base-line specification with time-invariant parameters and no macroeconomic variables.

Figure 6: Term Premia Estimates - Model 1



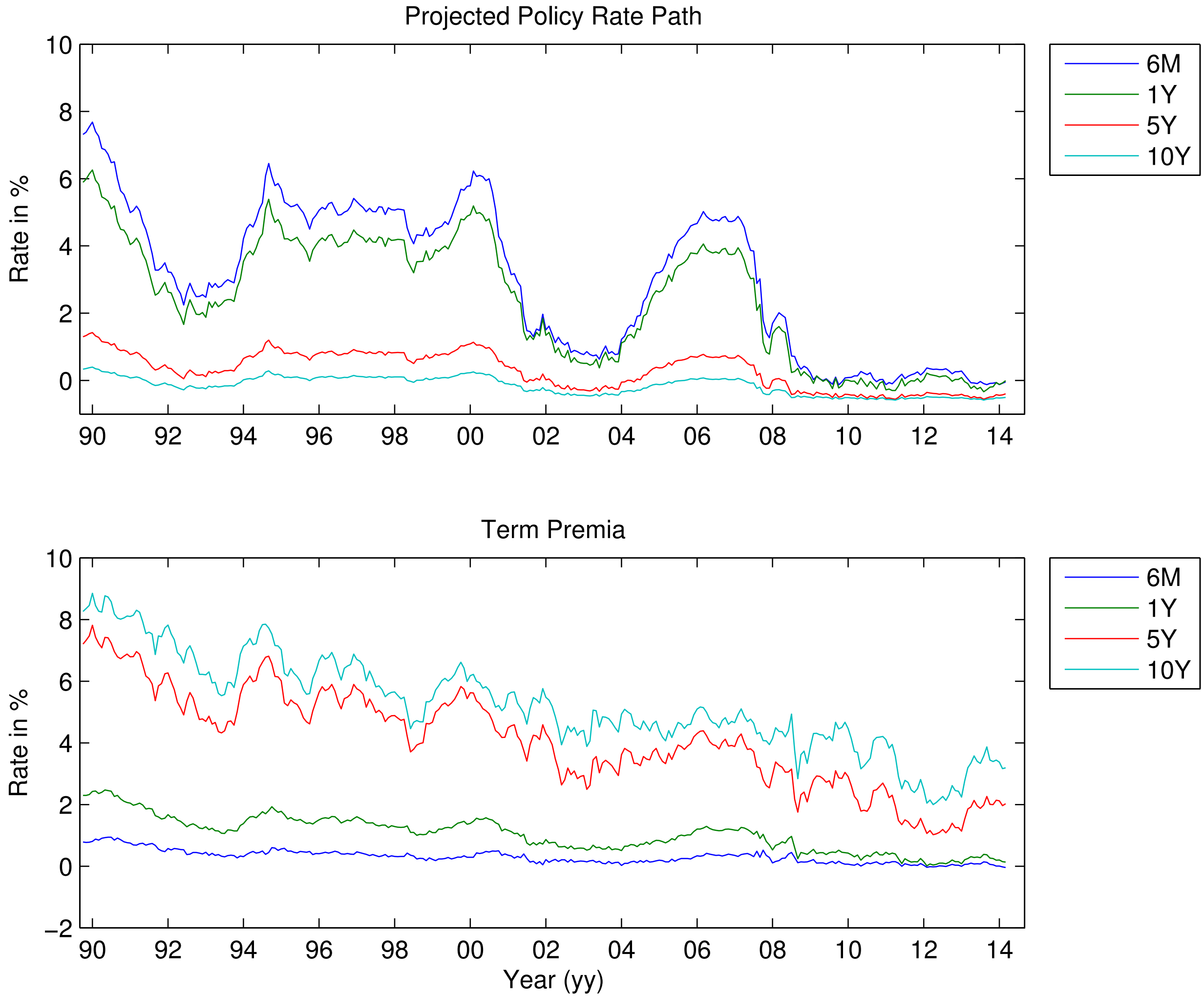
The upper panel shows the evolution over time of the projected path for the policy rate at the 6 months, 1 year, 5 year and 10 years projection horizons. At a given point in time, say in 2006 (06 on the x-axis of the panel) the lines show the expected location of the policy rate over the respective projection horizon. In other words, this panel depicts the accumulation of the short rate, using the model parameters, for the selected horizons of 6 months to 10 years. The lower panel shows the term premia for the 6 months, 1 year, 5 year and 10 years horizons, as defined by equation (16), which effectively is the difference between the model derived yield curve and the model projections made for the short rate. Results shown in the figure are derived from from the Rotated Dynamic Nelson-Siegel (RDNS) model following specification 2 on page 8: the base-line specification with time-invariant parameters and no macroeconomic variables.

Figure 7: Term Premia Estimates - Model 2



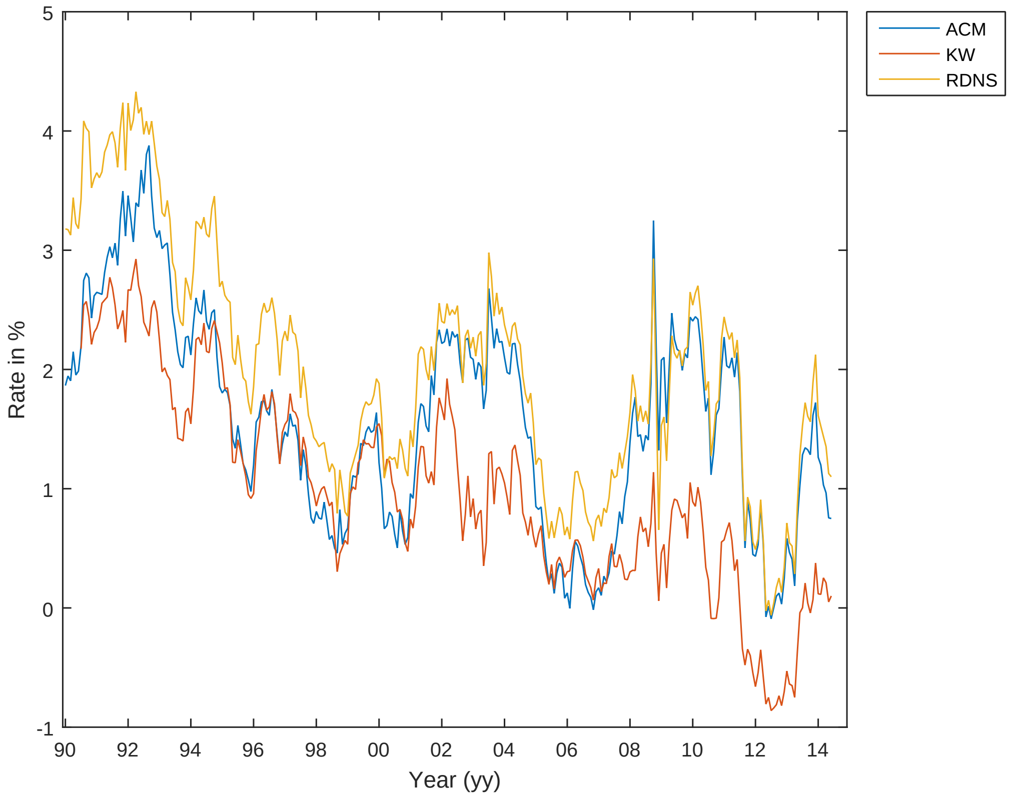
The upper panel shows the evolution over time of the projected path for the policy rate at the 6 months, 1 year, 5 year and 10 years projection horizons. At a given point in time, say in 2006 (06 on the x-axis of the panel) the lines show the expected location of the policy rate over the respective projection horizon. In other words, this panel depicts the accumulation of the short rate, using the model parameters, for the selected horizons of 6 months to 10 years. The lower panel shows the term premia for the 6 months, 1 year, 5 year and 10 years horizons, as defined by equation (16), which effectively is the difference between the model derived yield curve and the model projections made for the short rate. Results shown in the figure are derived from from the Rotated Dynamic Nelson-Siegel (RDNS) model following specification 3 on page 8: the base-line specification with time-invariant parameters and no macroeconomic variables.

Figure 8: Term Premia Estimates - Model 3



The upper panel shows the evolution over time of the projected path for the policy rate at the 6 months, 1 year, 5 year and 10 years projection horizons. At a given point in time, say in 2006 (06 on the x-axis of the panel) the lines show the expected location of the policy rate over the respective projection horizon. In other words, this panel depicts the accumulation of the short rate, using the model parameters, for the selected horizons of 6 months to 10 years. The lower panel shows the term premia for the 6 months, 1 year, 5 year and 10 years horizons, as defined by equation (16), which effectively is the difference between the model derived yield curve and the model projections made for the short rate. Results shown in the figure are derived from from the Rotated Dynamic Nelson-Siegel (RDNS) model following specification 4 on page 8: the base-line specification with time-invariant parameters and no macroeconomic variables.

Figure 9: Term Premia Estimates - Model 4



The evolution of 10 year term premia are shown for the period covering January 1990 to June 2014. The term premium from the RDNS model is calculated as described in (16) and the unconditional mean of the short rate process is constrained to equal the historical average of the maturity matched yield, i.e. the yield observed at the three month segment. The other two premia estimates are generated by Adrian, Crump, and Mönch (2013) and Kim and Wright (2005), as available on the internet.

Figure 10: Term Premia Comparison

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