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Asset allocation and risk taking under different interest rate regimes

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Abstract

We study the effects of low short-term interest rates on the optimal portfolio allocation in *Markowitz* portfolios and *Risk parity* portfolios. We propose a measure of *Portfolio Instability*, gauging the amount of optimal portfolio shifts needed to respond to exogenous shocks to the expected risk and return of the risky portfolio assets. *Portfolio Instability*, i.e. the selling pressure on riskier asset holdings, is found to be stronger the lower the risk-free interest rate. Heightened portfolio instability in the presence of low rates is found to emerge through two channels both of which incentivise the build-up of large and leveraged risky asset shares during calm periods which need to be unwound in the event of higher market volatility: first, low rates (mechanically) augment the excess return to be gained by investing in riskier assets and second, they are found to dampen volatility of riskier assets in the portfolio. The inverse relationship between portfolio instability and the risk-free rates is found to increase the closer the risk-free rate approaches the effective lower bound. Counterfactual analyses of the behaviour of optimal multi-asset portfolios demonstrate that the sell-off in riskier asset classes during the Covid crisis in March 2020 was more severe than would have been in the presence of higher short-term interest rates.

JEL Codes: C58, E52, G11, G12

Keywords: CAPM, Counterfactual analysis, portfolio optimization

Non-technical summary

Owing to various structural macro-economic factors weighing on the natural rate of interest, low short-term interest rates had, throughout approximately a decade, become a steady condition in financial markets of most advanced economies. By elevating riskier asset prices, low rates cheapened the internal and external financing conditions of non-financial companies. Moreover, low rates incentivised financial institutions to engage in search-for-yield behaviour, thereby adding risk to the asset side of their balance sheets. This search-for-yield behaviour has been widely studied for banks and is increasingly also studied for the non-bank financial sector. In this paper, we take the perspective of portfolio-optimising speculative multi-asset investors to study the implications of low interest rates on financial stability. We do so by gauging the extent to which they build-up of risky portfolio positions, including leveraged ones, during calm periods and the extent to which they subsequently re-allocate away from riskier assets into cash in the event of higher volatility, thereby amplifying the downward pressure on prices of riskier assets in the presence of adverse shocks to market returns and volatility. To achieve that, we propose a measure, coined *Portfolio Instability*, that gauges the aggregate shifts in the portfolio shares of *Markowitz* optimising investors, incurred by standardized shocks to asset price volatility under different interest rate regimes. As expected, portfolio instability rises with the share of risky assets which would have to be sold to an increasing amount if an external shock hampers their risk-return profile. Thereby, low rates favour higher risky asset shares via two channels: first, by mechanically widening the excess return offered by risky assets and second, by lowering risky asset volatility, thereby further ameliorating the risk/return profile of risky assets. We provide empirical evidence from a GARCH-M model that also the latter channel is at play.

We then investigate the economic significance of the amplifying effect of low risk-free rates in a market sell-off in a semi-natural experiment. To achieve that, we measure the amount of asset sales (in share terms) in multi-asset Markowitz and risk parity portfolios, incurred by the volatility shock posed by the Covid pandemic in March 2020 using historical benchmark price data of representative asset classes (equity, government and corporate bonds). Mirroring the experience of many speculative investment funds during that period, the simulated optimal portfolios featured initial levels of leverage and large subsequent outflows when the Covid shock arrived. Conversely, in counterfactual scenarios with higher initial levels of interest rates, initial leverage ratios of and subsequent risky asset sales from the optimal portfolios are significantly smaller. Thereby, the maginitude of the effects are inversely and non-linearly related to the interest rate level, implying that portfolio instability in such portfolios increases disproportionately as interest rate approach the effective lower bound of interest rates. Overall our results suggest, that the low level of interest rate contributed substantially to the severity in the market swings observed during that period.

We conclude that the low rate environment provides incentives for high levels of risk-taking, in principle by all portfolio optimising investors, but in particular by leveraged investment funds as these positions have to be suddenly unwound to preserve an optimal asset allocation. In fact, these dynamics might have amplified the market turmoil that was triggered by the Covid pandemic.

1 Introduction

The level of short-term risk-free rates has implications for risk premia of financial assets, on balance sheets of financial intermediaries and ultimately on financial stability. Low interest rates invite raising the leverage on financial institutions' balance sheets and increased investment in riskier financial assets, thereby lowering their risk premia and reducing their volatility (Drechsler et al., 2018). But as investors reach for yield during extended spells of low rates, their portfolios are increasingly prone to shocks as volatility spikes. As investors de-leverage and reduce their riskier positions - that is, as they run for the exit - they exacerbate the asset price deflation and inflict even higher volatility on their portfolios. Figure 1 documents this noticeable drop in riskier positions held by euro area based investors.

This paper provides evidence that a simple Markowitz portfolio (Markowitz, 1952) is able to replicate this very mechanism. We propose an indicator of portfolio instability measuring the variability of optimal portfolio shares in response to external shocks. We show empirically that low rates boost excess returns but also reduce asset price volatility, thereby raising optimal leverage ratios and risky asset shares in the portfolio. As a result, these portfolios are noticeably more sensitive to changes in the model parameters (expected mean return and variance) in the

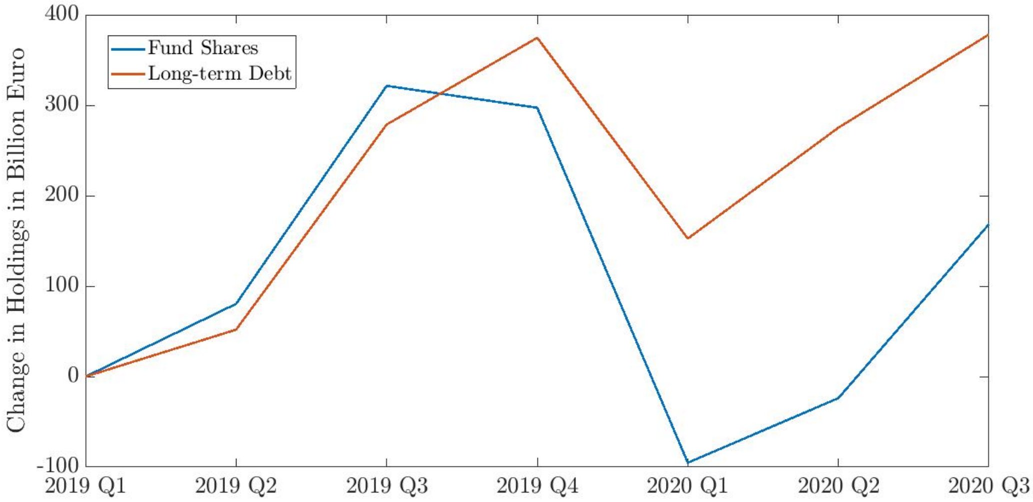

Figure 1: Quarterly aggregate evolution of Fund shares and Long-term debt in the Euro Area investors' portfolios. Amounts are in billions of euros, as retrieved from the ECB's Security and Holdings Statistics - by Sector (SHSS) database.

presence of low interest rates. Moreover, we provide a real-world example of our model prediction showing how portfolio shares were affected by the Covid shock as opposed to a counterfactual scenario in which the initial interest rate level was higher.

Finally, we extend our analysis to *Risk-Parity portfolios* introduced in Maillard et al. (2010), an increasingly popular approach to risk diversification in the portfolio selection. This approach is more robust to estimation errors in multi-asset portfolios (Cesarone et al., 2020). The results suggest that the interest rate level impact on portfolio instability is not limited to the mean-variance portfolio, but it is important also for investors who engage in volatility targeting strategies. In the appendix, we show that the results are robust across different regions.

This paper contributes to the a vast literature that studies the effects of low rates on risk-taking in (financial) asset markets. Low rates are not only conducive to a lower price of risk for equity (Bernanke and Kuttner, 2005; Laine, 2020), corporate credit markets (Gertler and Karadi, 2015) and real assets (Hanson and Stein, 2015), but their dampening effect on volatility additionally boosts the attractiveness of riskier assets from a mean/variance viewpoint. Drechsler et al. (2018) show that as the financial sector increases investments in riskier positions, volatility increases in the long run as shocks trigger larger sales of riskier assets than in a high interest rate world. Campbell (1987) and Glosten et al. (1993) provided empirical evidence of this causal link between interest rates and volatility empirically. Finally, our findings confirm empirical research on investors' portfolio decisions in response to expansionary monetary policy shocks as one of the drivers behind low interest rates. Evidence by Cecchetti et al. (2017) suggests that borrowing by banks *and* insurance companies increases with the length of the period of monetary easing. As for asset managers, Giuzio et al. (2021) show how investment fund investors reshuffle parts of their portfolios towards higher yielding and less liquid asset classes following a monetary easing shock, seeding the risk of more violent market swings in the advent of adverse shocks (Bubeck et al. (2018)) note that most of the portfolio shifts towards riskier assets occur passively, that is via out-performance of riskier assets over safer assets within investors' portfolios after a monetary easing shock.. Hau and Lai (2016) provide related evidence from a cross-country perspective, noting that shifts from safer money market fund shares to riskier equity fund shares are particularly evident in countries with low real interest rates. Also insurance companies increase the risk profile of their assets when funding conditions are loose (Becker and Ivashina, 2015) and increasingly venture into highly illiquid asset classes, including real estate or private equity (Fache Rousová and Giuzio, 2019).

The remainder of the paper is organized as follows: Section 2 introduces the theoretical framework and defines the *portfolio instability* (PI) measure as the volume of net asset transactions triggered by changes in the investors' expectations. Section 3 studies the sensitivity of portfolio instability to changes in the risk-free rate, in particular through its effects on risky asset volatility. Section 4 provides estimates of economic effects by applying the model to a conceivable risk-parity optimisiation during the recent Covid-19 crisis. Section 5 concludes.

2 Framework

We consider a universe of K assets and an investor who allocates his/her portfolio solving, at any time t, the asset allocation problem $\omega_t^* = f(\theta_t)$. θ_t is the conditioning information set available to the investor at time t and ω_t^* is the K vector of portfolio weights. Henceforth, we use θ instead of θ_t for simplicity of notation. In the seminal paper of Markowitz (1952), the mean variance investor is introduced assuming that returns are normally distributed and

$$\omega^* = \underset{\omega \in \mathbf{R}^K}{\arg\min} \quad \frac{1}{2}\gamma\omega'\Sigma\omega - \left[\omega'\tilde{\mu} + (1-\omega'\mathbf{1})r^f\right] \tag{1}$$

where $\mathbf{1}$ is the K-vector of all ones, r_t^f is the risk-free rate, $\tilde{\mu}$ is the K vector of expected returns and Σ is the $K \times K$ covariance matrix. γ is known as risk-aversion parameter. The higher γ, the lower is the investor's risk tolerance and the lower is the risk of the optimal portfolio. The optimization problem (1) has a closed form solution given by

$$\omega_t^* = \frac{1}{\gamma}\Sigma^{-1}\mu \tag{2}$$

where $\mu = \tilde{\mu} - r^f$ is the vector of *expected excess return*. The portfolio in equation (2) is known as tangency portfolio. The mutual fund theorem of Merton (1972) shows that the optimal portfolio is a linear combination of the tangency portfolio (2) and the risk-free asset.

In the Markowitz portfolio, the vector of relevant information needed to build the optimal portfolio is $\theta = (\mu, \text{vech}(\Sigma_t), \gamma)'$[1].

2.1 Portfolio Instability

In the rest of the section, we study the instability of investors portfolios with respect to changes in the input variables θ. We define the *Portfolio Instability* (PI) as a measure to evaluate the amount needed to re-balance the portfolio in the face of a shock. When expectations change, the investor has to re-balance his/her portfolio and may thereby also amplify market turbulence or liquidity shortages in order to remain optimally allocated. Let us consider θ^{pre} as the vector of relevant information before the advent of a shock and θ^{post} as the vector of relevant information after the shock. The external shock to the information vector is defined as $v = \theta^{\text{post}} - \theta^{\text{pre}}$. Portfolio instability is then defined as the directional derivative of the optimal portfolio $\omega^*(\theta^{\text{pre}})$

[1] The half-vectorization operator vech converts a matrix into a column vector. Specifically, the half-vectorization of a symmetric $K \times K$ matrix A is the $K(K+1)/2 \times 1$ column vector obtained by vectorizing only the lower triangular part of A

with respect to the shock v;

$$\mathrm{PI}_v(\theta^{\mathrm{pre}}) = D_v\omega(\theta^{\mathrm{pre}}) = \lim_{h\to 0}\frac{\omega(\theta^{\mathrm{pre}} + hv) - \omega(\theta^{\mathrm{pre}})}{h} =^{*} D\omega(\theta^{\mathrm{pre}})\cdot v \in \mathbf{R}^K \tag{3}$$

where $D\omega(\theta)$ is the Jacobian and * holds because the solution $\omega(\theta)$ is differentiable. It's worth noting that $D_v\omega(\theta^{\mathrm{pre}})$ is a K dimensional vector where the entry i represents the volume of the portfolio reallocation in the asset i. The higher the absolute value of the element i, the higher is the instability of the portfolio weight ω_i^* to the shock v. The element of v bears a positive (negative) sign for buy (sell) transactions. We define the *Total portfolio Instability* (TPI) as the ℓ_2-norm of the vector $\mathrm{PI}_v(\theta)$

$$\mathrm{TPI}_v(\theta) = ||\mathrm{PI}_v(\theta)||_2 = \sqrt{\sum_{i=1}^{K}(\mathrm{PI}_v(\theta)(i))^2}. \tag{4}$$

The higher the value of $\mathrm{TPI}_v(\theta)$, the higher the turnover induced by the shock v and the higher is the possible instability that the shock v will cause in the investor's portfolio. Annex A provides a practical example of this framework in a universe with two risky assets.

3 The role of short-term risk-free rates for portfolio instability

In this section, we study the sensitivity of portfolio weights and the Total Portfolio Instability metric with respect to changes in the risk-free rate for the Markowitz allocation (1). Thanks to the mutual fund theorem, we may restrict our analysis to two assets: one risky asset and one risk-free asset. We denote with σ_t^2 and r_t the variance and the expected returns of the risky asset and with r_t^f the risk-free rate with $r_t > r_t^f$. In this framework, the effect of the interest rate change on TPI and portfolio shares occurs through three channels: i) the risk-free rate's direct impact on volatility, ii) its direct impact on the excess return (of the risky asset) and iii) its indirect impact the excess return via volatility. We denote the key relations, with assumptions on the signs, as follow

$$
\begin{aligned}
\frac{\mathrm{d}\sigma_t^2}{\mathrm{d}r_t^f} &= \lambda_t > 0 \\
\frac{\mathrm{d}(r_t - r_t^f)}{\mathrm{d}r_t^f} &= \psi_t. \\
\frac{\mathrm{d}(r_t - r_t^f)}{\mathrm{d}\sigma_t^2} &= \mu_M > 0.
\end{aligned} \tag{5}
$$

The first line of equation (5) denotes the relation between the expected conditional variance of the risky asset and the risk-free rate. In turn. a higher variance augments the excess returns any risk-averse investor demands from the risky asset (third line). The second line is the more interesting. Mechanically, a lower risk-free rate augments the excess return of the risky asset ($\psi_t < 0$). However, a lower risk-free rate may also lower the excess return ($\psi_t > 0$) through its dampening effect on volatility, or the quantity of risk, ($\lambda_t > 0$) which in turn lowers the risk premium ($\mu_M > 0$). As both alternatives are plausible, we do not make any assumption on the sign of ψ_t and test for the relationship in the data. Similarly, the sign of μ_M is ambiguous. On the one hand, several studies show a significant positive risk-return relation (e.g. Scruggs 1998; Bali and Peng 2006; Chiang et al. 2015). On the other hand, negative risk-return relation is found (e.g. Campbell 1987; Brandt and Kang 2004; Ang et al. 2009). Other papers find contradictory results. For instance, Glosten et al. (1993) finds a positive sign of μ_M which becomes negative once the policy rate rate is included in the GARCH equation.

The investor optimal portfolio (2) with only one risky asset reduces to

$$
\begin{aligned}
\omega_t^1 &= \frac{(r_t - r_t^f)}{\gamma\sigma_t^2} \\
\omega_t^2 &= 1 - \omega_t^1,
\end{aligned} \tag{6}
$$

where ω_t^1 and ω_t^2 are the portfolio shares of the risky asset and the risk-free asset, respectively. The time index t will henceforth be omitted for simplicity of notation. The optimal portfolio share of the risky asset coincides with the sensitivity of the risky asset's portfolio weight (ω^1) with respect to the risk-free rate level, our first quantity of interest: a lower (short-term) interest rate level should increase the attractiveness and hence the portfolio weight of the risky asset.

$$\frac{\mathrm{d}\omega^1}{\mathrm{d}r^f} < 0 \iff \omega^1 > \frac{}{\gamma\lambda}. \tag{7}$$

A proof is provided in Appendix B.

A sufficient condition for (7) is $\lambda > 0$ and $\psi < 0$, that is that risky asset volatility declines and their excess return rises as the risk-free is lowered, thus making the risky asset more attractive from both a risk and return perspective.

The second quantity we are interested in is the sensitivity of the portfolio instability with respect to the risk-free rate level. For simplicity we separate the analysis for exogenous changes in the different parameters of $\theta = \left[r - r^f, \sigma^2, \gamma\right]$.

Proposition 1 *The sensitivities of the Total Portfolio Instability measure* (4) *with respect to the interest rate for the different types of exogenous shocks are*

$$\frac{dTPI_{[v^{ret},0,0]}}{dr^f} = -\sqrt{2}\frac{\lambda|v^{ret}|}{\gamma(\sigma^2)^2}. \tag{8}$$

$$\frac{dTPI_{[0,v^{var},0]}}{dr^f} = \begin{cases} \sqrt{2}\frac{\frac{d\omega^1}{dr^f}\sigma^2 - \lambda(\omega^1 - \frac{\mu_M}{\gamma})}{\gamma(\sigma^2)^2}|v^{var}|, & \textit{if } \omega^1 - \frac{\mu_M}{\gamma} > 0, \\ \sqrt{2}\frac{-\frac{d\omega^1}{dr^f}\sigma^2 - \lambda(\frac{\mu_M}{\gamma} - \omega^1)}{\gamma(\sigma^2)^2}|v^{var}|, & \textit{if } \omega^1 - \frac{\mu_M}{\gamma} < 0, \end{cases} \tag{9}$$

$$\frac{dTPI_{[0,0,v^{\gamma}]}}{dr^f} = \sqrt{2}\frac{\frac{d\omega^1}{dr^f}}{\gamma}|v^{\gamma}|. \tag{10}$$

The proof of the equation can be found in Appendix B. The signs associated to equations (8), (9) and (10) denotes the type of relation between the Total Portfolio Instability and the interest rate. If positive, then a low level of interest rate generates *ceteris paribus* a lower selling or buying pressure following a shock and the optimal portfolios are more robust. If negative, the optimal portfolio in a low rate environment is, *ceteris paribus*, more fragile to exogenous shocks. In the latter scenario, stability risks may arise as an unexpected event generates bigger re-balancing pressures and may exacerbate the liquidity and financial conditions.

The sign of equation (8) is negative under the first assumption in (5). The sign of the third equation strictly depends on the sign of equation (7), i.e. if condition (7) is fulfilled, then the sensitivity of the Total Portfolio Instability to the interest rate is negative. The second condition is less trivial because it is defined by case however, as we show in the following of the paper, in real-world applications we have $\omega^1 > \frac{\mu_M}{\gamma}$. If condition (7) is fulfilled, then a shock in the variance creates bigger market movements in the low rate environment. These findings postulate that when interest rates decline, investors respond by increasing the weight in the risky asset and decreasing the portfolio weight in the risk-free asset, which in turn raises the Total Portfolio Instability with respect to exogenous shocks in any of the model parameters. Notably, these results do not depend on the model used to evaluate the excess returns and volatility but they are intrinsic in the mean-variance optimization.

3.1 Market dynamic for mean-variance investors

In this section, we derive the dynamic of a market with an infinite number of mean-variance investors. In the previous section, we focus on the portfolio of a single agent but the purpose of this paper is to capture the dynamic of the system and the possible instabilities caused by a change in the market conditions.

Consider an infinite number of agents i with wealth W_i, risk-aversion γ_i and a market with K assets. In addition denote the total wealth as $W = \sum_{i=1}^{\infty} W_i$. For any investor the portfolio allocation is given by equation (6) as

$$\omega(i) = \frac{1}{\gamma_i}\Sigma^{-1}\mu. \tag{11}$$

The $K \times K$ covariance matrix Σ and the K vector of expected returns μ are equal across investors. Assuming a market clearing procedure, the aggregate demand of all the investors must be equal to the market portfolio multiplied by the total wealth, then

$$W\bar{\omega}^{\text{mrk}} = \sum_{i=1}^{\infty} W_i \frac{1}{\gamma_i}\Sigma^{-1}\mu \tag{12}$$

We use the notation $\bar{\omega}^{\text{mrk}}$ to distinguish the vector of the market portfolio allocation to the scalar ω^{mrk} which, in the other sections of the paper, represents the proportion of wealth allocated in the market portfolio. By definition, the weight of the market portfolio on asset i is given by the capitalization of asset i divided by the total market capitalization $\bar{\omega}^{\text{mrk}}$. Let $\hat{\mu}_M$ be the wealth weighted risk-aversion parameter and solve equation (12) for the excess expected returns, we get

$$\mu = \hat{\mu}_M \Sigma \bar{\omega}^{\text{mrk}}. \tag{13}$$

By definition, the market excess return is $\mu^{\text{mrk}} = \bar{\omega}^{\text{mrk}} \mu$ and the market variance by $\sigma^2_{\text{mrk}} = \bar{\omega}^{\text{mrk}\prime} \Sigma \bar{\omega}^{\text{mrk}}$, then

$$\mu_{\text{mrk}} = \hat{\mu}_M \bar{\omega}^{\text{mrk}\prime} \Sigma \bar{\omega}^{\text{mrk}} = \hat{\mu}_M \sigma^2_{\text{mrk}} \tag{14}$$

which is the CAPM formula.

Next, we test for the assumptions in (5) by means of a GARCH-M model, assuming different CAPM dynamics of the market portfolio.

3.2 Econometric model

In this section we test the excess return dynamic (14) using GARCH-M models with exogenous factors. We deploy the econometric model introduced in Glosten et al. (1993) and Scruggs (1998) to test the hypotheses put forth in the previous section. In particular, we explore different model specifications in which we estimate the parameters from equations (7), (8), (9) and (10).

The tests are conducted on weekly European market data spanning from January 1999 to October 2020. For the risk-free asset, we use the 1 week EONIA rate and the market portfolio is proxied by the broad EuroStoxx 600. The in-sample period spans from January 1999 to December 2019. The Covid-19 crisis period serves as the out-of-sample period. In Appendix C we present a similar study for the US and a longer time span (January 1952 to December 2019), showing that our main findings are robust across time and regions.

We start from the static CAPM model (henceforth *model 1.A*). This model perfectly fits in the mean-variance optimization framework as reported in Section 3.1 and we use the empirical specifications following Glosten et al. (1993), using a GARCH-M model. Recently, Kim and Kim (2016) and Campbell et al. (2018) used local volatility or stochastic volatility processes to estimate similar models. The expected excess return is given by

$$\begin{aligned} r_t - r_t^f &= \mu_0 + \mu_M \sigma_t^2 + \epsilon_t, \\ \sigma_t^2 &= Var_{t-1}(\epsilon_t) \\ \sigma_t^2 &= \omega + \alpha\epsilon_{t-1}^2 + \beta\sigma_{t-1}^2. \end{aligned} \tag{15}$$

Model in equation (15) is the baseline model where the risk-free rate does neither affect the market excess return nor the conditional variance. Several studies found a positive and significant relationship between the conditional variance and the excess return ($\mu_M > 0$). However, Glosten et al. (1993) found that μ_M becomes insignificant and sometimes negative if the risk-free rate is present in the conditional variance equation. We follow Glosten et al. (1993) in testing for this relationship with a GARCH-M model (henceforth *model 1.B*).

$$\begin{aligned} r_t - r_t^f &= \mu_0 + \mu_M \sigma_t^2 + \epsilon_t, \\ \sigma_t^2 &= Var_{t-1}(\epsilon_t) \\ \sigma_t^2 &= \omega + \alpha\epsilon_{t-1}^2 + \beta\sigma_{t-1}^2 + \tilde{\lambda} r_t^f. \end{aligned} \tag{16}$$

In (16) the risk-free rate indirectly affects excess return via its effect on the conditional variance ($\mu_M\tilde{\lambda}$). Empirically, we find a positive relationship between the risk-free rate and the conditional variance ($\tilde{\lambda} > 0$) [2]. According to the static CAPM model, there exists a positive relationship between the conditional variance and the excess return reflecting investors' risk aversion ($\mu_M > 0$). These two findings imply that model $1.B$ exhibits a positive relationship between the risk-free rate and the excess return[3].

Finally, we are aiming for more clarity about the role of the risk-free rate by allowing it to

[2] Glosten et al. (1993) finds an intuitive explanation behind this relationship: *"The use of nominal interest rates in conditional variance models has some intuitive appeal. It has been established by Fischer et al. (1981) that the variance of inflation increases with its level. To the extent that short-term nominal interest rates embody expectations about inflation, they could be a good predictor for future volatility in excess returns."*

[3] In turn,Laine (2020) finds an opposite sign for Europe during the Covid-19 crisis

directly affect excess returns in *model 1.C*.

$$\begin{aligned} r_t - r_t^f &= \mu_0 + \mu_M \sigma_t^2 + \mu_f r_t^f + \epsilon_t, \\ \sigma_t^2 &= Var_{t-1}(\epsilon_t) \\ \sigma_t^2 &= \omega + \alpha \epsilon_{t-1}^2 + \beta \sigma_{t-1}^2 + \tilde{\lambda} r_t^f. \end{aligned} \tag{17}$$

To summarise, models 1.A, B and C make the following assumptions on the effects of the risk-free rate on conditional variance and on the excess return. It relates with equation (5) as the following:

$$\begin{aligned} \text{model 1.A} \quad & \frac{\mathrm{d}\sigma_t^2}{\mathrm{d}r_t^f} = \lambda = 0 \\ & \frac{\mathrm{d}(r - r_t^f)}{\mathrm{d}r_t^f} = \psi = 0 \\ & \frac{\mathrm{d}(r_t - r_t^f)}{\mathrm{d}\sigma_t^2} = \mu_M. \\ \text{model 1.B} \quad & \frac{\mathrm{d}\sigma_t^2}{\mathrm{d}r_t^f} = \lambda = \tilde{\lambda} \\ & \frac{\mathrm{d}(r - r_t^f)}{\mathrm{d}r_t^f} = \psi = \mu_M \tilde{\lambda} \\ & \frac{\mathrm{d}(r_t - r_t^f)}{\mathrm{d}\sigma_t^2} = \mu_M. \\ \text{model 1.C} \quad & \frac{\mathrm{d}\sigma_t^2}{\mathrm{d}r_t^f} = \lambda = \tilde{\lambda} \\ & \frac{\mathrm{d}(r - r_t^f)}{\mathrm{d}r_t^f} = \psi = \mu_M \tilde{\lambda} + \mu_f \\ & \frac{\mathrm{d}(r_t - r_t^f)}{\mathrm{d}\sigma_t^2} = \mu_M \end{aligned} \tag{18}$$

.

where $\tilde{\lambda}$ is estimated in model 1.A, B and C, μ_M is estimated in model 1.B and C and μ_f is estimated in model 1.C. Table 1 reports the corresponding econometric estimates of the parameters of models 1.A, 1.B and 1.C. In model 1.A the relationship between the conditional variance and the excess return is positive and slightly significant ($\mu_M = 2.319$, t-stat = 1.690). In model 1.B, the risk-free rate is found to have a positive and slightly significant impact on the conditional variance ($\tilde{\lambda} = 0.034$, t-stat 1.638) as the impact of the conditional variance on the excess return decreases ($\mu_M = 1.970$, t-stat $= 1.398$). As we explained, this behaviour may be due to the indirect impact of the interest rate in the excess return and it was previously observed in Glosten et al. (1993) and Scruggs (1998). In model 1.C where we separately consider the impact of the risk-free rate in the conditional variance and the expected return, $\tilde{\lambda}$ remains positive and marginally significant. The direct impact of the risk-free rate on the excess return is negative but not significant ($\mu_f = -3.739$ and t-stat -1.497) but the relationship between the conditional variance and the risk-free rate becomes highly significant ($\mu_M = 3.023$ and t-stat 1.914).

Considering the lower part of the table, model 1.B and 1.C are in line with our assumptions on the negative sign of the quantities (7), (8), (9) and (10). In model 1.C, the sufficient condition for a negative relationship between the risk-free rate on the risky asset portfolio weight ($\psi < 0$) is satisfied since ψ is negative and $\tilde{\lambda}$ is positive. Model 1.B has a positive ψ and the sufficient condition is not satisfied. Yet, when evaluating (7), (8), (9) and (10) at each point in time, it is found that total portfolio instability tends to rise with declining levels in the risk-free rate.

In the last row of the table we show the AIC (Aikake Information Criterion). Model 1.A is the worst to capture the dynamic of the data, model 1.B ranks second while model 1.C is the best performing.

The difference in terms of fit from model 1.B and model 1.C is very small. If we consider the AIC, model 1.C is preferred over model 1.B, however the Likelihood Ratio Test does not reject the null of model 1.B with a p-value of 0.06. For this reason, the results of both models 1.B and model 1.C are examined. We evaluate the Total Portfolio Instability of investor (1) over the horizon January 1999-December 2019. We assume an investor with a risk aversion parameter of $\gamma = 15$, and set the exogenous shocks to $v^{\text{ret}} = -5\%$ (i.e. a decrease in the excess return by 5 percentage points per annum), $v^{\text{var}} = 30\%$ (i.e. an increase in the annualised volatility by 30 percentage points) and $v^{\gamma} = 0.2\%$ (i.e. an increment in the risk aversion parameter by 0.2 percentage points which, in our case, means that the investors become more risk-averse) . Figure 2 shows the evolution of the Total Portfolio Instability with respect to the two models. The results are filtered with a year rolling window average. Using both models, we record a steady increase in portfolio instability in recent years. At the face of the results of the previous section,

Parameters	model 1.A	model 1.B	model 1.C
μ_0	0.001	0.001	0.002
	(1.213)	(1.560)	(1.882)
μ_M	2.319	1.970	3.023
	(1.690)	(1.398)	(1.914)
μ_f			-3.739
			(-1.497)
$\omega \times 10^5$	2.724	1.816	1.917
	(2.632)	(1.950)	(2.019)
α	0.193	0.182	0.183
	(2.809)	(2.874)	(3.041)
β	0.780	0.773	0.768
	(14.703)	(13.439)	(14.421)
$\tilde{\lambda}$		0.034	0.036
		(1.638)	(1.728)
		0.067	-3.629
		(1.018)	(-1.465)
$\frac{\mathrm{d}\omega^1}{\mathrm{d}r^f}$		< 0	< 0
$\frac{\mathrm{d}TPI_{[v^{\mathrm{ret}},0,0]}}{\mathrm{d}r^f}$		< 0	< 0
$\frac{\mathrm{d}TPI_{[0,v^{\mathrm{var}},0]}}{\mathrm{d}r^f}$		< 0	< 0
$\frac{\mathrm{d}TPI_{[0,0,v^{\gamma}]}}{\mathrm{d}r^f}$		< 0	< 0
AIC	-5154.7	-5160.9	$\mathbf{-5162.3}$

Table 1: Estimation of models 1.A, 1.B and 1.C in equations (16) and (17) and model implied sensitivities to the interest rate. In the first section we report the estimates and the Robust t ratios in parenthesis. The estimates ψ is evaluated according to (5) and the Robust t-statistic is evaluated with the delta method. In the second section we report the different quantities in equations (7), (8), (9) and (10). When the sign is < 0 the partial derivative is negative in the whole sample, when the sign is > 0 the partial derivative is positive and when the sign is $\pm$ the partial derivative can be both positive and negative. In the last row we show the AIC and the best performing model according to this measure is bolded.

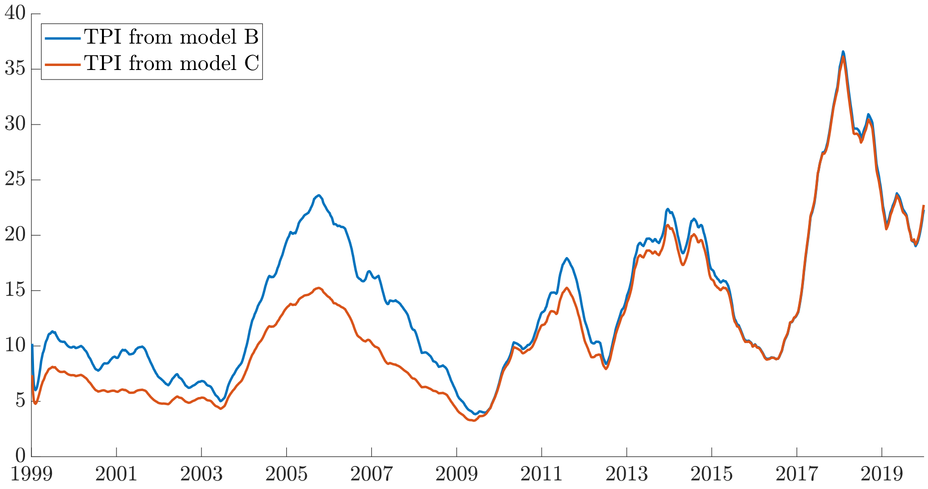

Figure 2: Total Portfolio Instability evaluated from model 1.B (16) and model 1.C (17). The measures are filtered using a 1-year moving window and assuming an impact of 30% in the variance and -5% in the expected excess return.

the secular decline in both the risk-free rates and in market volatility have set the conditions for total portfolio instability to rise to the highest level ever recorded.

3.3 Non linear behaviour of the Total Portfolio Instability

In the previous two sections, we showed theoretically and empirically that portfolio shifts in reactions to shocks are more pronounced in the presence of low (vs. high) rates. Indeed, owing to the non-linear properties of the TPI function, this relationship is convex with disproportionately high levels of portfolio instability for rates around or even below zero. Figure 3 depicts this non-linearity for the estimated parameters in table 1 for models 1.B and 1.C.

3.4 Covid-19 crisis and the Markowitz investor. A counterfactual analysis

In this section, we deploy the estimated parameters from Section 3.2 to characterise the role of the low rates in the recent Covid-19 crisis. Our findings suggest that the pre-Covid, low interest rate level have likely amplified the market turmoil in March 2020. We quantify by how much fewer portfolio weights would have moved in models in the presence of higher interest rates in the framework presented by model 1.B (16) and model 1.C (17). We assume an investor solving the Markowitz problem in (6) as news about the risk-adjusted return of risky assets arrive $(\epsilon_t/\sqrt{\sigma_t^2})$.

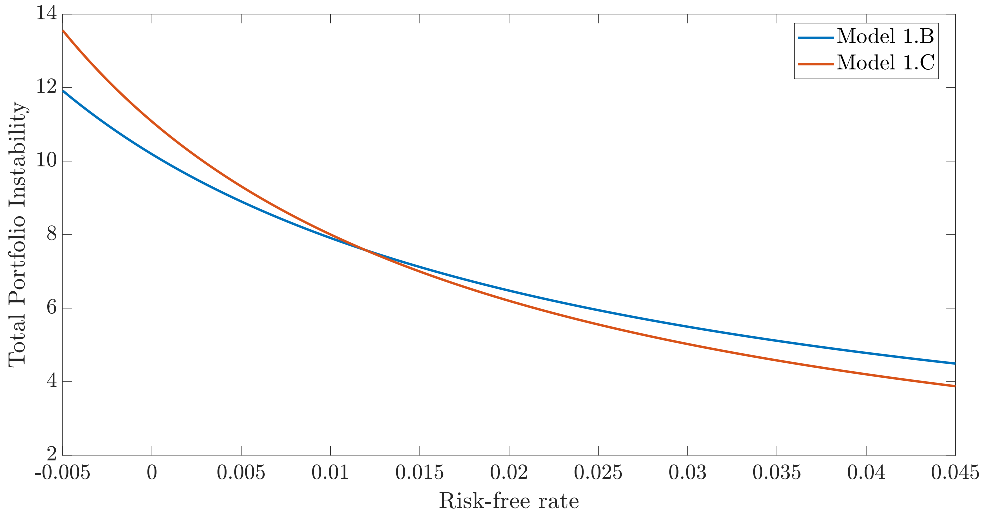

Figure 3: Total Portfolio Instability sensitivity evaluated from model 1.B (16) and model 1.C (17) for different values of the risk-free rate. For the calculations, we use the unconditional levels of σ^2 and $r - r^f$ given by models (16) and (17) for different levels of r^f.

We do not consider the risk-free rate as a source of the shock but rather as a model parameter affecting the portfolio choices in real-time. As a first step, we use the observed EONIA rate to model a baseline scenario in which investors optimise their portfolios in response to changes in the expected return and variance profile of the various asset classes induced by the Covid-19 pandemic.

Using model (16) and (17), we forecast the expected excess return and variance and, as a residual we get the out of sample residuals ϵ_t^{OOS}. In a second step, we create a counterfactual, augmented EONIA path. We denote with $r_t^{f,\text{CF}}$ as the counterfactual risk-free rate and calculate the corresponding expected excess return and variance as

$$\begin{aligned} E_{t-1}\left[(r_t - r_t^f)^{\text{CF}}\right] &= \mu_0 + \mu_M \sigma_t^{2,\text{CF}} + \mu_f r_t^{f,\text{CF}} \\ \sigma_t^{2,\text{CF}} &= \omega + \alpha\left(\epsilon_{t-1}^{\text{OOS}}\right)^2 + \beta\sigma_{t-1}^2 + \tilde{\lambda} r_t^{f,\text{CF}}, \end{aligned} \tag{19}$$

with portfolio weights ω^{CF} and using the explicit portfolio solution (6). Note that we set $\mu_f = 0$ for model 1.B. As a counterfactual, we use a risk-free rate that is one percentage point higher than the observed EONIA rate. In our comparative analysis, we scale the portfolio weights to 0 at the beginning of January 2020 so that we can trace the relative portfolio shifts in the course of 2020.

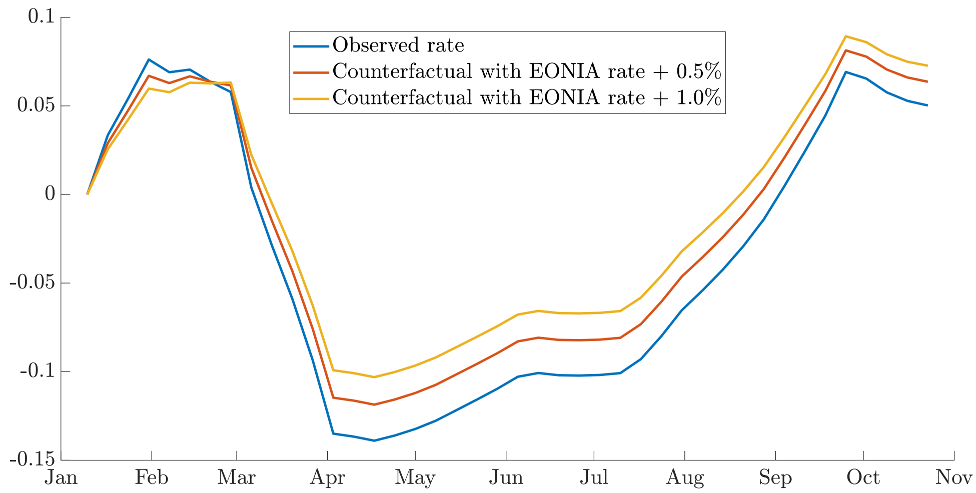

Figure 4: Change in the portfolio share since January 2020 given actual risk-free rates (in blue) and counterfactual risk-free rates (orange: EONIA + 0.5%, yellow: EONIA + 1%) in model 1.B (16). The measures are filtered using a 1-month moving window.

Figure 4 shows the evolution of weights under model 1.B, while figure 5 shows the evolution under model 1.C. Their dynamics are very similar. The reduction in the equity share during March was at the order of -15% of the portfolio size prevailing in January 2020. The sale of equity would have been less pronounced if the initial risk-free rate had been higher (50 basis points: -12.5%, 100 basis points: -10%).

4 The Risk Parity investor

The Markowitz portfolio, inspected in the previous section, has two important short-comings which make them little representative of investors active in financial markets. First, optimal Markowitz multi-asset portfolios typically allocate most of the wealth to only a few assets. Second, Markowitz weights are very sensitive to estimation errors whereas small changes in the expected returns and variance estimation lead to very different optimal allocations (see Best and Grauer (1991) and Chopra and Ziemba (1993)). To counter these drawbacks, a new class of portfolio optimisation is considered, that relies solely on the relative risk profile of the various asset classes. Cesarone et al. (2020) shows that portfolios built using risk diversification strategies are the most robust to noise. In particular, they find that *Risk Parity Portfolios*,

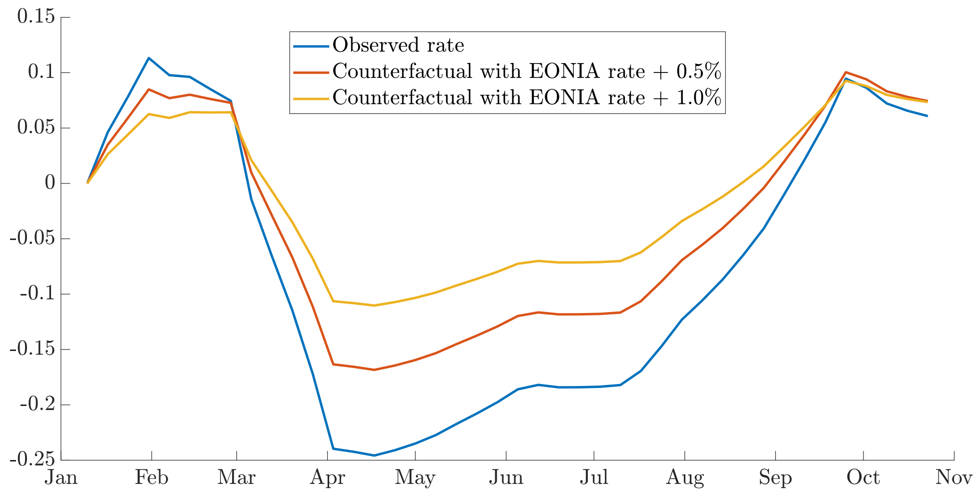

Figure 5: Observed sell-off (in blue) and counterfactual sell-off (in orange with 0.5% and yellow with 1%) evaluated with model 1.C (17). The measures are filtered using a one month moving window.

introduced by Maillard et al. (2010), are more stable with respect to estimation errors. Before the Covid-19 crisis, an estimated USD 300 billion was invested in funds following risk-parity strategies. The mere size of this market segment may render the portfolio flows induced by external shocks macro-critical. Portfolio shifts may thus be able to amplify the initial shock, e.g. by putting further downward pressure on the prices of riskier assets and/or raising market volatility. In this section, we use the Risk Parity portfolio to robustify the results presented in previous sections.

The Risk Parity Portfolio (RP) aims to spread the risk among all assets equally and each asset contributes to the total portfolio volatility, denoted as $\sigma(\omega_t)$, equally. More precisely, the volatility contribution of asset i should be equal to the volatility contribution of asset j, $\omega_t^i \frac{\partial\sigma(\omega_t)}{\partial\omega_t^i} = \omega_t^j \frac{\partial\sigma(\omega_t)}{\partial\omega_t^j}$ for any asset i and j and any time t. That is,

$$\omega_t^i \frac{\partial\sigma(\omega_t)}{\partial\omega_t^i} = \omega_t^i \frac{(\Sigma_t \omega_t)_i}{\sigma(\omega_t)}. \tag{20}$$

At time $t-1$, the RP investor with a variance-covariance matrix expectation $\hat{\Sigma}_t$, re-balances his/her portfolio weights ω_t solving the optimization problem:

$$\omega_t = \arg\min_{\omega} \sum_{i=i}^{K} \left[\frac{\sigma(\omega)}{K} - \omega^i \frac{(\Sigma_t \omega)_i}{\sigma(\omega)} \right]^2 \tag{21}$$

The portfolio obtained minimizing (21) is not well identified; indeed it has infinite solutions. As per the Markowitz portfolio (1) we can identify the optimal portfolio imposing that $\mathbf{1}\omega = 1$. However, we impose the volatility targeting constraint $\sigma(\omega) = \omega^*$. A key advantage of the volatility targeting constraint is that we can explore another dimension of the investor behaviour, where he/she increases leverage during low volatility periods and decreases the leverage during high volatility periods. The final optimization problem then becomes

$$\begin{aligned} \omega_t = \arg\min_{\omega} \quad & \sum_{i=i}^{K} \left[\frac{\sigma(\omega)}{K} - \omega_i \frac{(\Sigma_t \omega)_i}{\sigma(\omega)} \right]^2 \\ \text{subject to} \quad & \sigma(\omega) = \sigma^* \end{aligned} \tag{22}$$

There is no closed-form solution for the risk parity portfolio. Hence, there is *a priori* no mathematical relationship between the behaviour of the portfolio weights and the risk-free interest rate. In the following empirical exercise, we estimate the portfolio weights and the Portfolio Instability by means of numerical optimization and derivation.

4.1 Econometric model and Portfolio Instability

We consider a risk-parity investor with a target of 8% annual volatility and a universe of 3 risky assets, namely the equity (EUROSTOXX 600), the long-term government bond (Markit iBoxx Sovereigns Eurozone Index) and the Corporate bond (Bloomberg Barclays Euro Aggregate Corporate Total Return Index).

Portfolio weights in a risk parity portfolio are set such that the every asset class in the portfolio contributes the same share to the overall portfolio volatility. Hence, the optimal weights depend on the level of the target volatility. We assume that the risk-parity investor allocates 50% of her risk in the equity and the 50% in the fixed income. Of the latter, 25% is allocated in the government bonds and corporate bonds, respectively. The investor forecasts the future covariance matrix using a Dynamic Conditional Correlation model (Engle, 2002)

$$\begin{aligned} r_{i,t} - r_t^f &= \mu_i + \epsilon_{i,t} \\ \epsilon_{i,t} &= \sigma_{i,t} z_t, \quad E[z_t] = 0,\ E[z_t^2] = 1 \\ \sigma_{i,t}^2 &= \omega_i + \alpha_i \epsilon_{i,t-1}^2 + \beta_i \sigma_{i,t-1}^2 \\ V_t &= D_t R_t D_t \\ R_t &= \Delta_t^{-1} Q_t \Delta_t^{-1} \\ Q_t &= \bar{Q}(1 - \alpha_{DCC} - \beta_{DCC}) + \alpha_{DCC}(\epsilon_t \epsilon_t') + \beta_{DCC} Q_{t-1}. \end{aligned} \tag{23}$$

where $\bar{Q}$ is the unconditional mean of the pseudo-correlation matrix Q, D_t is the diagnoal matrix of conditional volatilities, and $\Delta_t = \text{diag}(Q_t)$ is the matrix with only the diagonal elements of the pseudo-correlation matrix. From now on we refer to model (23) as *Model 2.A*.

In the spirit of Section 3.2, we allow the risk-free rate to affect the volatility dynamics. For this reason, we propose an alternative model

$$\begin{aligned} r_{i,t} - r_t^f &= \mu_i + \epsilon_{i,t} \\ \epsilon_{i,t} &= \sigma_{i,t} z_t, \quad E[z_t] = 0,\ E[z_t^2] = 1 \\ \sigma_{i,t}^2 &= \omega_i + \alpha_i \epsilon_{i,t-1}^2 + \beta_i \sigma_{i,t-1}^2 + \lambda_i r_t^f \\ V_t &= D_t R_t D_t \\ R_t &= \Delta_t^{-1} Q_t \Delta_t^{-1} \\ Q_t &= \bar{Q}(1 - \alpha_{DCC} - \beta_{DCC}) + \alpha_{DCC}(\epsilon_t \epsilon_t') + \beta_{DCC} Q_{t-1}, \end{aligned} \tag{24}$$

where λ_i captures the dependency between the risk-free rate and the volatility level. We denote this model as *Model 2.B*.

Table 2 shows the estimated parameters from model (23) and (24). The AIC shows the better fit given by model 2.B. In addition, the likelihood ratio test rejects the null hypothesis of model 2.A with a p-value of 10^{-9}. The sensitivity of the volatility with respect to the risk-free level is positive and marginally significant for the equity and the corporate bonds while it is negative and non-significant for government bonds. Low interest rates tend to mute the volatility in the two riskiest assets (including equity and corporate bonds) but increase the volatility levels

Parameters	Model 2.A			Model 2.B		
	Equity	Gov. Bond	Corp. Bond	Equity	Gov. Bond	Corp. Bond
$\omega \times 10^5$	2.480	0.183	0.024	1.617	0.193	0.017
	(2.547)	(2.819)	(1.755)	(1.868)	(2.493)	(1.357)
α	0.176	0.092	0.091	0.165	0.093	0.082
	(3.060)	(3.634)	(3.700)	(3.242)	(3.681)	(3.345)
β	0.797	0.844	0.897	0.791	0.843	0.895
	(18.245)	(22.715)	(41.963)	(17.074)	(22.854)	(37.262)
$\lambda \times 10^2$				3.432	−0.021	0.047
				(1.707)	(−0.421)	(1.980)
α_{DCC}		0.039			0.038	
		(10.049)			(10.003)	
β_{DCC}		0.959			0.961	
		(240.477)			(252.157)	
AIC		−24218.4			**−24257.4**	

Table 2: Estimation of models 2.A and 2.B in equations (23) and (24). Any column represents a different asset class. In parenthesis we report the t-statistics In the last row we show the AIC and the best performing model according to this measure is bolded.

of government bonds consistent with the notion of convexity in bond pricing. As in Section 3.3, we evaluate the behaviour of the total portfolio instability with respect to the risk-free rate. We define a range of possible risk-free rate levels ranging from -0.5% to 4.5%. In the GARCH-DCC-type model (24) the unconditional expected values are defined as

$$\begin{aligned} \bar{\sigma}_i^2(r_f^*) &= \frac{\omega_i + \lambda_i r_f^*}{1 - \alpha_i - \beta_i} \\ \bar{V} &= \bar{D}\bar{R}\bar{D}, \end{aligned} \tag{25}$$

where $\bar{D}$ is the diagonal matrix with the $\bar{\sigma}_i$ element on the i-th diagonal.

Using equation (25), we can study the (T)PI of the risk-parity portfolio for varying levels of the risk-free rate. We study the portfolio shifts induced by an exogenous 20 percentage point increase in volatility increase in all the asset classes and a parallel increase in all correlation coefficients by 10 percentage points. Figure 6 reports the results. The behaviour of the Total Portfolio Instability is qualitatively similar to the Total Portfolio Instability in figure 2. Portfolio Instability increases dis-proportionately as interest rates decline. Concerning the Portfolio Instability, the most susceptible asset class are corporate bonds whereas the share of government bonds is almost indifferent with respect to the interest rate level (as suggested by the estimated parameter λ in table 2). In addition, we see that the investor is susceptible to higher flight-to-safety behaviour for low level of the risk-free rate. That results are similar to the Markowitz

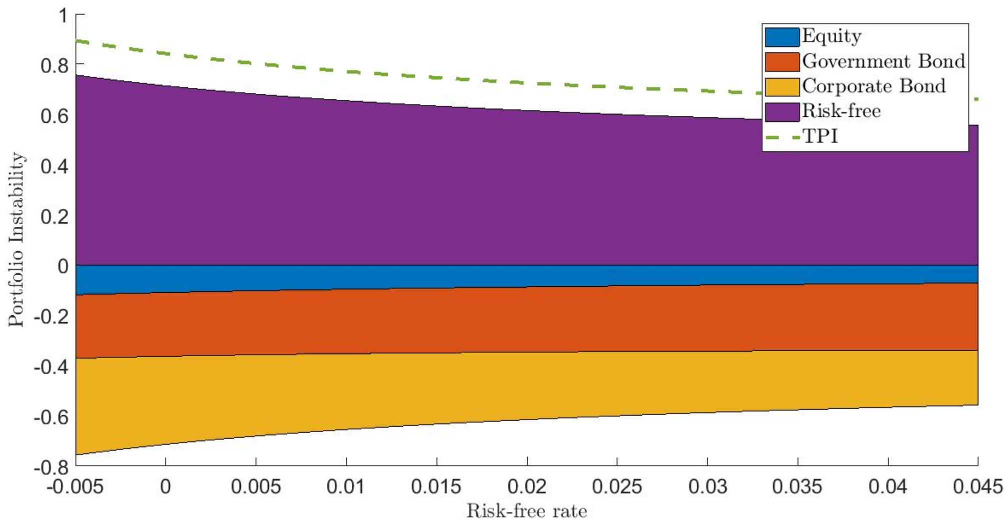

Figure 6: Total portfolio instability of the risk parity portfolio given different levels of the interest rate. In blue we report the TPI for the volatility shock scenario, in orange we report the TPI for the correlation shock scenario

case suggests that our findings are robust across different investor optimization strategies.

4.2 Covid-19 crisis and the Risk parity investor. A counterfactual analysis

The counterfactual analysis, equivalent to 3.4, is based on the counterfactual rates $r_t^{f,\text{counter}}$ of 50 and 100 basis points higher than the one observed in the EONIA rate. Figure 7 shows the results for a risk-parity investor during the Covid-19 crisis, with a volatility target of 8% p.a. Qualitatively the result confirm what we observe in figure 4 and 5 for the Markowitz portfolio, but quantitatively the results differ by some margin. The risk-parity investor with an 8% volatility target deploys high levels of leverage in a period of low volatility, like the one before the Covid-19 pandemic. This high level of leverage translates into high portfolio re-balancing volumes during the crisis. In the baseline scenario, the investor sells almost 200% of the portfolio value of risky assets, reflecting large previous short-positions in cash (paying the short-term interest rate). This would decrease to 170% and 150% in the counterfactual scenario with a higher EONIA rate [4]. Given that the risky portfolio is composed of 3 asset classes, we can assess the selling pressure on every component. Figure 8 shows the sell-off decomposed into the different components. In all three asset classes, a higher risk-free rate produces a smaller

[4]For details, see Vassallo et al. (2020)

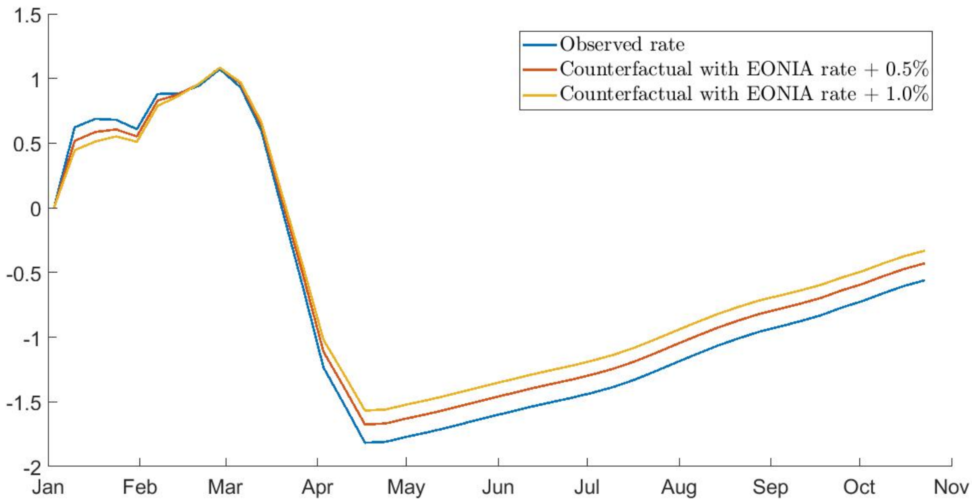

Figure 7: Observed sell-off (in blue) and counterfactual sell-off (in orange with 0.5% and yellow with 1%) evaluated from model 2.B (24). The measures are filtered using a one month moving window.

sell-off in those asset classes with a negative relation to the risk-free rate and volatility (namely equity and corporate bonds). By contrast, the government bonds portion of the portfolio proves to be more robust to differences in the risk-free level, as already suggested by its λ parameter in table 2.

5 Conclusions

This paper investigates the role of the low interest rates on optimal portfolio allocations following shocks to asset price returns and risk. We show that a low level of interest rate increases the exposures to risky assets through the build-up of highly leveraged positions. The risk-taking behaviour of investors constitutes a risk to financial stability that we measure as *Portfolio Instability*: the amount of asset sales/purchases needed to re-optimise the portfolio when shocks occur. Heightened levels *Portfolio Instability* can lead to large and sudden portfolio shifts in the event of exogenous shocks to the portfolio parameters (mean and variance). We demonstrate in scenario exercises that portfolio re-balancing flows in Markowitz and risk parity portfolios were significantly larger during the Covid-19 crisis than would have been the case in the presence of higher interest rates.

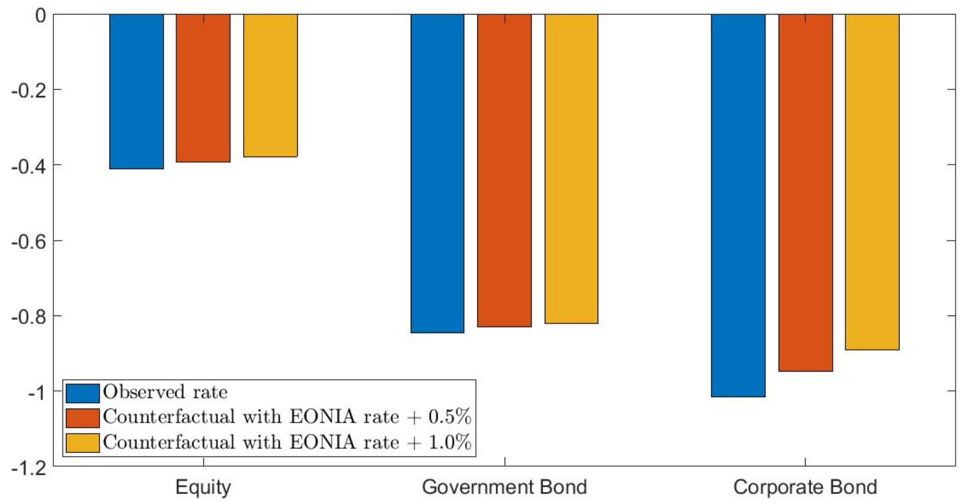

Figure 8: Observed sell-off (in blue) and counterfactual sell-off (in orange with 0.5% and yellow with 1%) evaluated from model 1.C (17). The measures are filtered using a one month moving window.

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Appendix

A Example of portfolio instability with two risky assets

The following example provides some economic intuition behind the definition of Portfolio Instability introduced in section 2. Consider a market with two risky assets A and B where the expected annual variance of asset A is 4%, the expected annual variance of asset B is 1% and the expected correlation between the two asset is 0.2, which leads to an expected co-variance of 0.4%. Let us consider that the initial expectation for the annual returns are 6.5% for asset A and 4% for asset B. An investor allocates his/her portfolio by solving the Markowitz problem (1) with $\gamma = 5$. The optimal portfolio is composed of shares of 26% in asset A and the 74% in asset B. After an external shock, the investor revises down the expected returns to 4.5% for asset A and 3% for asset B. Summing up, the initial value of θ was

$$\theta^{\text{pre-shock}} = \left[\mu_A, \mu_B, \sigma_A^2, \sigma_B^2, \sigma_{A,B}, \gamma\right] = [6.5\%, 4\%, 4\%, 0.4\%, 1\%, 5]$$

and after a shock to expectations amounting to

$$v^{\text{shock}} = [-0.02, -0.01, 0, 0, 0, 0]$$

the investor finds herself in a new state of the world which is characterised by

$$\theta^{\text{post-shock}} = \theta^{\text{pre-shock}} + v^{\text{shock}} = [4.5\%, 3\%, 4\%, 0.4\%, 1\%, 5]\,.$$

In the new state of the world $\theta^{\text{post-shock}}$, the optimal asset allocation is given by investing the 21% of the portfolio into asset A and the 79% in asset B. That corresponds to a reduction of 5 percentage points in asset A which are being reinvested in asset B. Figure 9 shows the optimal allocation surface which represents the investment in asset A for different values of expected returns. The red dot shows the optimal allocation in the $\theta^{\text{pre-shock}}$ environment while the blue dot shows the new optimal allocation in the state of the world $\theta^{\text{post-shock}}$. The blue line which connects the two dots is the line generated by the directional derivative. The total portfolio

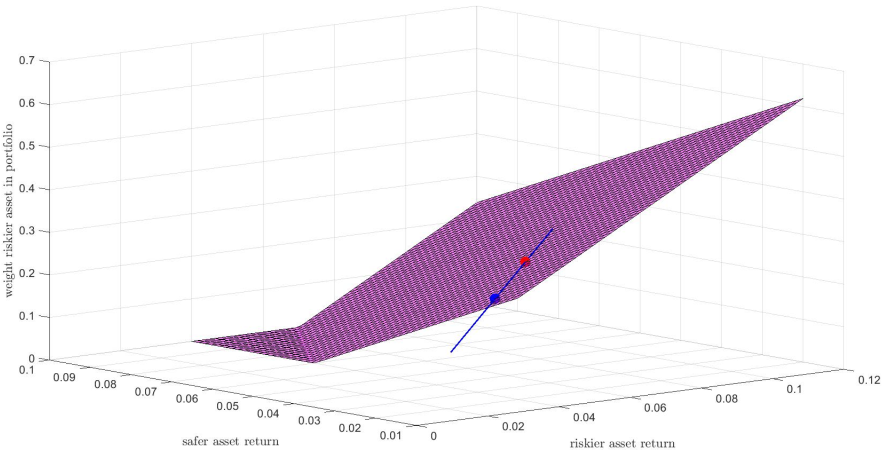

Figure 9: Directional derivative. In the z-axis we report the weights of the riskier asset A according to the expected returns of the two assets. The x-axis and y-axis show the expected returns of asset B and asset A respectively.

instability is the length of the segment between the two dots. The length of the segment depends on the shape of the optimal allocation surface. In this example, $\mathrm{TPI}_{v^{\text{shock}}}(\theta^{\text{pre-shock}}) = 0.067$.

B Proof Propositions 7 and 1

The proof of proposition 7 is trivial. Indeed the value of the total derivative is given by

$$\frac{\mathrm{d}\omega^1}{\mathrm{d}r^f} = \frac{\gamma\psi\sigma^2 - \lambda\gamma(r - r^f)}{(\gamma\sigma^2)^2} \tag{26}$$

Using equation (6) we get that

$$\frac{\mathrm{d}\omega^1}{\mathrm{d}r^f} = \frac{}{\gamma\sigma^2} - \frac{\lambda\gamma}{\sigma^2\gamma}\omega^1 < 0 \iff \omega^1 > \frac{}{\gamma\lambda} \tag{27}$$

which concludes the proof.

For Proposition 1, using the information set $\theta = \left[r - r^f, \sigma^2, \gamma\right]$ for the investor problem (6), we analytically evaluate the Portfolio Instability measure defined in (3). The Jacobian matrix reduces to

$$
\begin{aligned}
D\omega(\theta) &= \begin{bmatrix} \frac{\partial \omega^1}{\partial (r-r^f)} & \frac{\partial \omega^1}{\partial \sigma^2} & \frac{\partial \omega^1}{\partial \gamma} \\ \frac{\partial \omega^2}{\partial (r-r^f)} & \frac{\partial \omega^2}{\partial \sigma^2} & \frac{\partial \omega^2}{\partial \gamma} \end{bmatrix} = \begin{bmatrix} \frac{\partial \omega^1}{\partial (r-r^f)} & \frac{\partial \omega^1}{\partial \sigma^2} & \frac{\partial \omega^1}{\partial \gamma} \\ -\frac{\partial \omega^1}{\partial (r-r^f)} & -\frac{\partial \omega^1}{\partial \sigma^2} & -\frac{\partial \omega^1}{\partial \gamma} \end{bmatrix} \\
&= \begin{bmatrix} \frac{1}{\gamma\sigma^2}, & \frac{\mu_M \sigma^2-(r-r^f)}{(\gamma\sigma^2)^2}, & -\frac{(r-r^f)}{\gamma^2\sigma^2} \\ -\frac{1}{\gamma\sigma^2}, & -\frac{\mu_M \sigma^2-(r-r^f)}{(\gamma\sigma^2)^2}, & \frac{(r-r^f)}{\gamma^2\sigma^2} \end{bmatrix}
\end{aligned}
\tag{28}
$$

Firstly, we consider an exogenous change in the risky asset's expected excess return. We denote with v^{ret} the size of the exogenous shock, then

$$
PI_{[v^{\text{ret}},0,0]} = D\omega(\theta) \begin{bmatrix} v^{\text{ret}} \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{v^{\text{ret}}}{\gamma\sigma^2} \\ -\frac{v^{\text{ret}}}{\gamma\sigma^2} \end{bmatrix} \tag{29}
$$

and the TPI can be evaluated as

$$
TPI_{[v^{\text{ret}},0,0]} = \sqrt{2}\frac{|v^{\text{ret}}|}{\gamma\sigma^2}. \tag{30}
$$

The sensitivity of the TPI to the risk-free rate is given by

$$
\frac{\mathrm{d}TPI_{[v^{\text{ret}},0,0]}}{\mathrm{d}r^f} = -\sqrt{2}\frac{\lambda|v^{\text{ret}}|}{\gamma(\sigma^2)^2}. \tag{31}
$$

The second possible shock is a shock to the variance of the risky asset (v^{var}):

$$
PI_{[0,v^{\text{var}},0]} = D\omega(\theta) \begin{bmatrix} 0 \\ v^{\text{var}} \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{\mu_M\sigma^2-(r-r^f)}{(\gamma\sigma^2)^2} v^{\text{var}} \\ -\frac{\mu_M\sigma^2-(r-r^f)}{(\gamma\sigma^2)^2} v^{\text{var}} \end{bmatrix} = \begin{bmatrix} \frac{\frac{\mu_M}{\gamma}-\omega^1}{(\sigma^2)} v^{\text{var}} \\ -\frac{\frac{\mu_M}{\gamma}-\omega^1}{(\gamma\sigma^2)} v^{\text{var}} \end{bmatrix} \tag{32}
$$

The total portfolio instability is given by $TPI_{[0,v^{\text{var}},0]} = \sqrt{2}\frac{|\omega^1-\frac{\mu_M}{\gamma}|}{(\gamma\sigma^2)}|v^{\text{var}}|$ and the sensitivity with respect to the risk-free rate

$$\frac{\mathrm{d}TPI_{[0,v^{\mathrm{var}},0]}}{\mathrm{d}r^f}\begin{cases}\sqrt{2}\frac{\frac{\partial\omega^1}{\partial r^f}\sigma^2-\lambda(\omega^1-\frac{\mu_M}{\gamma})}{\gamma(\sigma^2)^2}|v^{\mathrm{var}}|, & \text{if } \omega^1-\frac{\mu_M}{\gamma}>0,\\ \sqrt{2}\frac{-\frac{\partial\omega^1}{\partial r^f}\sigma^2-\lambda(\frac{\mu_M}{\gamma}-\omega^1)}{\gamma(\sigma^2)^2}|v^{\mathrm{var}}|, & \text{if } \omega^1-\frac{\mu_M}{\gamma}<0,\end{cases} \tag{33}$$

For the third possible shock in the investor preference, i.e. in the parameter γ, we find similar results as in the previous scenario

$$PI_{[0,0,v^\gamma]}=D\omega(\theta)\begin{bmatrix}0\\0\\v^\gamma\end{bmatrix}=\begin{bmatrix}-\frac{(r-r^f)}{\gamma^2\sigma^2}v^\gamma\\ \frac{(r-r^f)}{\gamma^2\sigma^2}v^\gamma\end{bmatrix} \tag{34}$$

$$TPI_{[0,0,v^\gamma]}=\sqrt{2}\frac{\omega^1}{\gamma}|v^\gamma|. \tag{35}$$

$$\frac{\partial TPI_{[0,0,v^\gamma]}}{\partial r^f}=\sqrt{2}\frac{\frac{\partial\omega^1}{\partial r^f}}{\gamma}|v^\gamma|<0\iff\frac{\partial\omega^1}{\partial r^f}<0. \tag{36}$$

This concludes the proof.

C Econometric model with US Data

In this appendix we test the theoretical findings in Section 3 using US monthly data from April 1951 to December 2019. The market proxy is the value-weighted portfolio provided in the French's website[5] and risk-free rate proxy is the one-month T-bill rate. Table 3 shows the estimates for the models 1.A (15), 1.B (16) and 1.C (17). The results are consistent with the findings in table 1 with an higher significance, partly due to the longer time series and partly due to the lower frequency of the data. The only difference between the US estimates and the EU estimates is the sign of the parameter ψ for the model 1.B but in both cases, the estimates are not significant.

Figure 10 shows the same non-linear behaviour observed in figure 3. The results show that the portfolio instability behaviour is robust across countries.

[5]Kenneth R. French website.

Parameters	model 1.A	model 1.B	model 1.C
μ_0	0.003	0.007	0.009
	(1.037)	(1.837)	(2.862)
μ_M	1.931	-0.209	3.497
	(1.185)	(-0.090)	(1.703)
μ_f			-2.666
			(-3.420)
$\omega \times 10^5$	10.096	5.414	5.034
	(2.271)	(1.367)	(1.106)
α	0.115	0.115	0.114
	(3.730)	(3.027)	(2.969)
β	0.837	0.795	0.798
	(20.167)	(11.639)	(12.551)
$\tilde{\lambda}$		0.029	0.028
		(2.272)	(2.480)
		-0.006	-2.569
		(-0.089)	(-3.455)
$\frac{\partial \omega^1}{\partial r^f}$		< 0	< 0
$\frac{\partial TPI_{[v^{\text{ret}},0,0]}}{\partial r^f}$		< 0	< 0
$\frac{\partial TPI_{[0,v^{\text{var}},0]}}{\partial r^f}$		< 0	< 0
$\frac{\partial TPI_{[0,0,v^{\gamma}]}}{\partial r^f}$		< 0	< 0
AIC	-2841.5	-2854.1	$-$**2868.2**

Table 3: Estimation of models 1.A, 1.B and 1.C in equations (16) and (17) and model implied sensitivities to the interest rate. In the first section we report the estimates and the Robust t ratios in parenthesis. The estimates ψ is evaluated according to (5) and the robust t-statistic is evaluated with the delta method. In the second section we report the different quantities in equations (7), (8), (9) and (10). When the sign is < 0 the partial derivative is negative in the whole sample, when the sign is > 0 the partial derivative is positive and when the sign is $\pm$ the partial derivative can be both positive and negative. In the last row we show the AIC and the best performing model according to this measure is bolded.

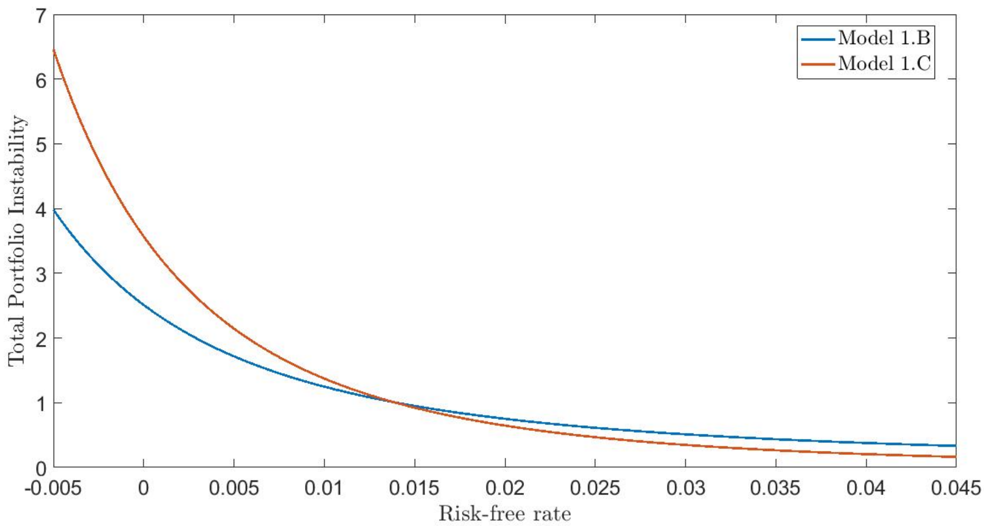

Figure 10: Total Portfolio Instability sensitivity evaluated from model 1.B (16) and model 1.C (17) for different values of the risk-free rate. For the calculations, we use the unconditional levels of σ^2 and $r - r^f$ given by models (16) and (17) for different levels of r^f.

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